Ordinary Differential Equation (ODE):

ODE is an equation consists of unknown Function and its derivatives. The unknown function depends on only one variable. **The order** of an ODE is the order of the highest derivative appearing in the equation.

→ For Example:

$$\frac{dy}{dx} + x = y \qquad 1^{st} \text{ order}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad 2^{nd} \text{ order}$$

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad 3^{rd} \text{ order}$$

y: dependent variable

x: independent variable

The degree: is the degree of highest derivative occurring on it

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0 \qquad 2^{nd} \text{ order } 1^{st} \text{ degree}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + y = 0 \qquad 2^{nd} \text{ order } 3^{rd} \text{ degree}$$

NoTe: $\frac{dy}{dx}$ or y' , $\frac{d^2y}{dx^2}$ or y'' , --- etc

ODE classified into:-

1. First order.
2. Second order.
3. high order.

« First Order ODE »

- Separable Equation
- Homogeneous Equation
- Exact Equation
- Linear Equation

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[1] Separable Equation:

The standard form is: $\frac{dy}{dx} = g(y) f(x)$

ويسمى طريقة فصل المتغيرات.

خطوات الحل :-

1. فصل المتغيرات مع مشتقاتها.
2. القانون الرئيسي :-

$$\boxed{\int g(y)\,dy = \int f(x)\,dx + c}$$

3

Ex: Solve $\dfrac{dy}{dx} = 2x(y^2+9)$

① فصل المتغيرات
② تكامل

$$\int \frac{dy}{(y^2+9)} = \int 2x\,dx$$

$$\boxed{\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)}$$

$$\frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right) = x^2 + c \quad]*3$$

$$\tan^{-1}\left(\frac{y}{3}\right) = 3x^2 + 3c$$

$$\frac{y}{3} = \tan(3x^2+3c)$$

$$y = 3\tan(3x^2+3c)$$

Ex: Solve $\dfrac{dy}{dx} = \dfrac{-x\cos x}{1-6y^5}$, $y(\pi)=0$

Sol/ $(1-6y^5)\,dy = (-x\cos x)\,dx$

$$\int (1-6y^5)\,dy = -\int (x\cos x)\,dx + c$$

$$\Rightarrow \int (1-6y^5)\,dy = y - y^6$$

$$\Rightarrow -\int (x\cos x)\,dx = -x\sin x + \int \sin x\,dx = -x\sin x - \cos x$$

$\int u\,dv = u\cdot v - \int v\cdot du$

$u = x$, $du = dx$

$dv = \cos x\,dx$, $v = \sin x$

$\therefore\ y - y^6 = -x\sin x - \cos x + c$

at $x = \pi$, $y = 0$

$$0 - 0 = -\cancel{\pi\sin\pi}^{\,0} + \cos\pi + c$$

$$0 = 0 - (-1) + c \Rightarrow 0 = 1 + c \Rightarrow \boxed{c = -1}$$

$\therefore$ The complete Solution is

$$y - y^6 = -x\sin x - \cos x + 1 \quad]\times -1$$

$$y^6 - y = x\sin x + \cos x + 1$$

ممكن

* If the 1st order has form $y' = f(ax+by+C)$, it can be reduced to separable eqn. as :- [Reducible to Separable]

$$\boxed{\text{Let } u = ax+by+c \;\rightarrow\; \frac{du}{dx} = a + b\frac{dy}{dx}}$$

١) فرض قيمة u ثم اشتقاقها.
٢) مساواة المشتقة مع المعادلة الاصلية.
٣) تطبيق قانون Sep.

Ex: Solve

$$\frac{dy}{dx} = (4x-y+1)^2$$

Sol: Let $u = 4x - y + 1$

$$\frac{du}{dx} = 4 - \frac{dy}{dx} \quad ①$$

$$\frac{dy}{dx} = 4 - \frac{du}{dx} \quad ②$$ → مساواتها مع المعادلة الاصلية

المعادلة الاصلية → $\frac{dy}{dx} = u^2$

$$\therefore \; u^2 = 4 - \frac{du}{dx}$$ → → هنا يمكن حلها بطريقة Sep. (فصل المتغيرات)

$$\frac{du}{dx} = 4 - u^2$$

$$\int \frac{du}{4-u^2} = \int dx + C \quad ③$$

$$\boxed{\int \frac{du}{a^2-u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)}$$

$$\frac{1}{2}\tanh^{-1}\left(\frac{u}{2}\right) = x + C$$

$$\tanh^{-1}\left(\frac{4x-y+1}{2}\right) = 2x + 2C$$

Ex: Solve $\quad x\dfrac{dy}{dx} = y + 4x^5\cos^2\left(\dfrac{y}{x}\right)$

نلاحظ السؤال لا يمكن فصل المتغيرات مباشرة، لذلك يجب فرض $\boxed{u}$

Let $u = \dfrac{y}{x}$

$$\frac{du}{dx} = \frac{1}{x}\frac{dy}{dx} - y\frac{1}{x^2} \longrightarrow$$

هنا يجب سحب $\dfrac{dy}{dx}$ لمساواتها مع المعادلة الاصلية

$$\left[\frac{du}{dx} + \frac{y}{x^2} = \frac{1}{x}\frac{dy}{dx}\right] * x$$

$$\boxed{x\frac{du}{dx} + \frac{y}{x} = \frac{dy}{dx}} \longrightarrow ①$$

المعادلة الاصلية $\left[x\dfrac{dy}{dx} = y + 4x^5\cos^2\left(\dfrac{y}{x}\right)\right] \div x$ للتخلص من $\underline{x}$

$$\frac{dy}{dx} = \frac{y}{x} + 4x^4\cos^2\left(\frac{y}{x}\right)$$ ← يمكن تعويض قيمة u

$$\boxed{\frac{dy}{dx} = u + 4x^4\cos^2(u)} \longrightarrow ②$$

الان يمكن مساواة المعادلة ① مع المعادلة ②

$$x\frac{du}{dx} + \cancel{u} = \cancel{u} + 4x^4\cos^2(u)$$

$$x\frac{du}{dx} = 4x^4\cos^2(u) \longrightarrow$$

اصبحت المعادلة جاهزة لحلها بطريقة Sep.

$$\int\frac{du}{\cos^2 u} = \int\frac{4x^4}{x}\,dx + C$$

$$\int \sec^2 u = \int 4x^3\,dx + C$$

$$\tan u = x^4 + C$$

$$\tan\left(\frac{y}{x}\right) = x^2 + C \Longrightarrow y = x\tan^{-1}(x^4 + C)$$

② Homogenouse D.E :-

المعادلات المتجانسة

The D.E $M(x,y)dx + N(x,y)dy$ is homogenouse if M and N are homogenouse function of the same degree.

خطوات الحل :-

١- وضع المعادلة في حالة $f(x,y) = \frac{dy}{dx}$

٢- جعل $f(x,y)$ في حاله $(\frac{y}{x})$.

٣- تعويض $v = \frac{y}{x}$

٤- القانون $\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$

٥- مساواة القانون مع المعادلة في السؤال

٦- تكامل الطرفين.

Ex: Solve $(x^2 - y^2)\,dy = (xy)\,dx$

Sol: $\frac{dy}{dx} = \frac{xy}{x^2 - y^2} \quad \} \div x^2$

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1 - \frac{y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{v}{1 - v^2}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

مساواة المعادلتين

$$v + x\frac{dv}{dx} = \frac{v}{1 - v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1 - v^2} - v \Rightarrow x\frac{dv}{dx} = \frac{v - v(1 - v^2)}{1 - v^2}$$

$$x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} + v^3}{1-v^2}$$

$$x\frac{dv}{dx} = \frac{v^3}{1-v^2}$$ $\Big\} \Rightarrow$ جعل كل مشتقة بطرف
ثم التكامل

$$\int \frac{1-v^2}{v^3}\,dv = \int \frac{1}{x}\,dx$$

$$\int \left(\frac{1}{v^3} - \frac{1}{v}\right) dv = \int \frac{1}{x}\,dx \Rightarrow \int v^{-3}\,dv - \int \frac{1}{v}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2v^2} - \ln v = \ln x + C$$

$$-\frac{1}{2}\left(\frac{x}{y}\right)^2 - \ln\left(\frac{y}{x}\right) = \ln x + C$$

Ex: solve/ $(y+x)\,dy + (x-y)\,dx = 0$

sol/ $(y+x)\,dy = -(x-y)\,dx$

$$\frac{dy}{dx} = \frac{y - x}{y+x} \quad \Big]\div x$$

$$\frac{dy}{dx} = \frac{y/x - 1}{y/x + 1} \quad , \text{Sub } v = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{v-1}{v+1} \quad , \boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\frac{v-1}{v+1} - v = x\frac{dv}{dx}$$

$$\frac{v-1-v(v+1)}{v+1} = x\frac{dv}{dx}$$

$$\frac{\cancel{v}-1-v^2-\cancel{v}}{v+1} = x\frac{dv}{dx} \Rightarrow \frac{-1-v^2}{v+1} = x\frac{dv}{dx}$$

$$-\frac{1+v^2}{1+v} = x\frac{dv}{dx}$$ } فصل المتغيرات

$$\int -\frac{1+v}{1+v^2}\,dv = \int \frac{dx}{x}$$

$$\int \frac{1}{1+v^2} + \frac{v}{1+v^2} = -\int \frac{dx}{x}$$

$$\tan^{-1} v + \frac{1}{2}\ln(1+v^2) = -\ln x + C$$

$$\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left(1+\left(\frac{y}{2}\right)^2\right) = -\ln x + C$$

Ex: Solve $x^2\frac{dy}{dx} = xy + y^2$ } $\div x^2$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$$, sub $v = \frac{y}{x}$

$$\frac{dy}{dx} = v + v^2$$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\cancel{v} + v^2 = \cancel{v} + x\frac{dv}{dx}$$

$$v^2 = x\frac{dv}{dx}$$ } طرفين * وسطين

$$\int \frac{dv}{v^2} = \int \frac{dx}{x}$$

$$\frac{-1}{v} = \ln x + C$$

$$-\frac{x}{y} = \ln x + C$$

Ex:

Solve: $\frac{dy}{dx} = \frac{2y^4 + x^4}{x y^3}$ $\} \div x^4$

$$\frac{dy}{dx} = \frac{2\left(\frac{y}{x}\right)^4 + 1}{\left(\frac{y}{x}\right)^3} = \frac{2v^4+1}{v^3}$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}} \Rightarrow \frac{2v^4+1}{v^3} = v + x\frac{dv}{dx}$$

$$\frac{2v^4+1}{v^3} - v = x\frac{dv}{dx}$$

$$\frac{2v^4+1-v(v^3)}{v^3} = x\frac{dv}{dx}$$

$$\frac{2v^4+1-v^4}{v^3} = x\frac{dv}{dx}$$

$$\frac{v^4+1}{v^3} = x\frac{dv}{dx} \quad \}$$ طرفين بوسطين

$$\int \frac{v^3}{v^4+1}\,dv = \int \frac{dx}{x} \quad \} \rightarrow$$

ملاحظة/من خصائص (ln) نوفر مشتقة المقام في البسط

$v^4 = 3v^3$

$$\frac{1}{4}\int \frac{4v^3}{v^4+1}\,dv = \int \frac{dx}{x}$$

$$\frac{1}{4}\ln(v^4+1) = \ln x + c$$

$$\frac{1}{4}\ln\left[\left(\frac{y}{x}\right)^4 + 1\right] = \ln x + c$$

③ Equation Reducible to Separable or Homogeneous:

Given the ODE of the form:- $\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$

*If $c_1 = c_2 = 0$, the Eq. is homogeneous.

*If $(c_1, c_2) \neq (0,0)$, the Eq not homo, this cases should be reduced to homo. as follows.

* Case 1

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

then put $Z = a_1 x + b_1 y$ and the eq. converted to Separable eq.

* Case 2

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

the put $x = X + h \Rightarrow dx = dX$

$y = Y + k \Rightarrow dy = dY$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{a_1 X + b_1 Y + a_1 h + b_1 k + c_1}{a_2 X + b_2 Y + a_2 h + b_2 k + c_2} \quad \text{------} (\#)$$

$a_1 h + b_1 k = -c_1$

$a_2 h + b_2 k = -c_2$

put $\begin{cases} a_1 h + b_1 k + c_1 = 0 \\ a_2 h + b_2 k + c_2 = 0 \end{cases}$ inordre to reduce eq. (#) to homogeneous.

$$\therefore h = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad k = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or

Now, by substituting the values of h and k in the given ODE (eq #), the ODE is reduced to homogeneous.

Ex: Solve: $\dfrac{dy}{dx} = \dfrac{2x+y-1}{4x+2y+5}$

$\Rightarrow * \ (c_1, c_2) \neq 0$ and it is not homo.

$\Rightarrow * \ \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4-4 = 0 \Rightarrow$ case 1

$\Rightarrow$ put $z = a_1 x + b_1 y$

$$\boxed{z = 2x + y} \Rightarrow \boxed{\frac{dz}{dx} = 2 + \frac{dy}{dx}} \quad \text{----} (\#)$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dz}{dx} - 2$

$\Rightarrow$ sub $\underline{z}$ value in the question eq.

نساوي $\frac{dy}{dx}$ بالمعادلتين

$$\therefore \frac{dy}{dx} = \frac{z-1}{2z+5} \quad \text{---- sub in } (\#) \text{ eq.}$$

$$\boxed{\frac{dz}{dx} = 2 + \frac{z-1}{2z+5}} \Rightarrow \frac{dz}{dx} = \frac{4z+10+z-1}{2z+5} = \frac{5z+9}{2z+5}$$

تنقلب

$$\left[\int \frac{2z+5}{5z+9}\, dz = \int dx \right] * \frac{5}{2}$$

$$\int \frac{10z+25}{10z+18}\, dz = \int \frac{5}{2}\, dx$$

$$\int \frac{10z+18+7}{10z+18}\, dz = \frac{5}{2}\int dx \Rightarrow \int \left(1 + \frac{7}{10z+18}\right) dz = \frac{5}{2}\int dx$$

$$z + \frac{7}{10}\ln(10z+18) = \frac{5}{2}x + c$$

$\Rightarrow$ z نعوض قيمة

$$(2x+y) + \frac{7}{10}\ln[10(2x+y)+18] = \frac{5}{2}x + c$$

Ex: Solve $\quad \frac{dy}{dx} = \frac{x+y-3}{x-y-1}$

$\Rightarrow (c_1, c_2) \neq 0 \longrightarrow$ not homo.

$\Rightarrow \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0 \rightsquigarrow$ case 2

$a_1x + b_1y + c_1$
$a_1(X+h) + b_1(Y+k) + c_1$
$a_1X + b_1Y + a_1h + b_1k + c_1$

$\therefore$ put $x = X + h \longrightarrow dx = dX$
$\quad\quad y = Y + k \longrightarrow dy = dY$

$$\Rightarrow \frac{dY}{dX} = \frac{a_1X + b_1Y + a_1h + b_1k + c_1}{a_2X + b_2Y + a_2h + b_2k + c_2}$$

$$\frac{dY}{dX} = \frac{X + Y + (h+k-3)}{X - Y + (h-k-1)} \Rightarrow \begin{cases} h+k=3 & (1) \\ h-k=1 & (2) \end{cases}$$

$h = 3-k$ — sub in (2)
$\to 3-k-k=1$
$2k = 3-1 \Rightarrow k=1$
$h = 3-1 \Rightarrow h=2$

$$\Rightarrow \boxed{\frac{dY}{dX} = \frac{X+Y}{X-Y}}$$ became homogeneous eq. $] \div X$

$$\frac{dY}{dX} = \frac{1 + Y/X}{1 - Y/X}, \quad v = \frac{Y}{X} \Rightarrow \frac{dY}{dX} = \frac{1+v}{1-v}$$

$$\frac{dY}{dX} = v + X\frac{dv}{dX} \Rightarrow \frac{1+v}{1-v} = v + X\frac{dv}{dX}$$

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dX}{X}$$

$$\int \left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \int \frac{dX}{X}$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln X + C$$

$$\tan^{-1}\left(\frac{Y}{X}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{Y}{X}\right)^2\right) = \ln X + C$$

$\begin{cases} X = x - h = x - 2 \\ Y = y - k = y - 1 \end{cases} \therefore \tan^{-1}\left(\frac{y-1}{x-2}\right) - \frac{1}{2}\left[1 + \left(\frac{y-1}{x-2}\right)^2\right] = \ln(x-2) + C$