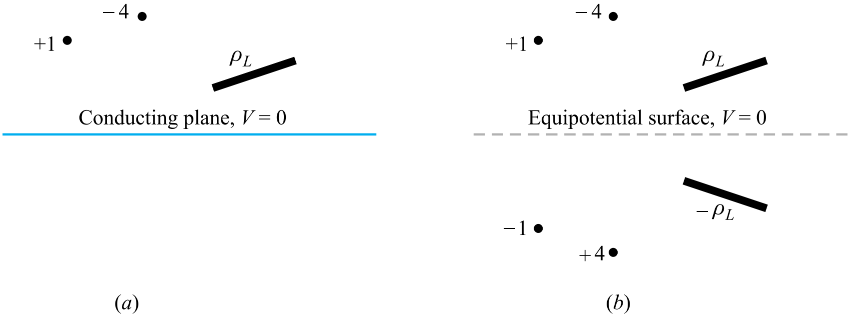

Figure 5.7 (*a*) A given charge configuration above an infinite conducting plane may be replaced by (*b*) the given charge configuration plus the image configuration, without the conducting plane.

charges. The radial vector from the positive line charge to P is $\mathbf{R}_+ = 2\mathbf{a}_x - 3\mathbf{a}_z$, while $\mathbf{R}_- = 2\mathbf{a}_x + 3\mathbf{a}_z$. Thus, the individual fields are

$$\mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+}\mathbf{a}_{R+} = \frac{30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}$$

and

$$\mathbf{E}_- = \frac{30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}$$

Adding these results, we have

$$\mathbf{E} = \frac{-180\times 10^{-9}\mathbf{a}_z}{2\pi\epsilon_0(13)} = -249\mathbf{a}_z \text{ V/m}$$

This then is the field at (or just above) P in both the configurations of Figure 5.8, and it is certainly satisfying to note that the field is normal to the conducting plane, as it must be. Thus, $\mathbf{D} = \epsilon_0\mathbf{E} = -2.20\mathbf{a}_z$ nC/m^2, and because this is directed *toward* the conducting plane, ρ_S is negative and has a value of -2.20 nC/m^2 at P.

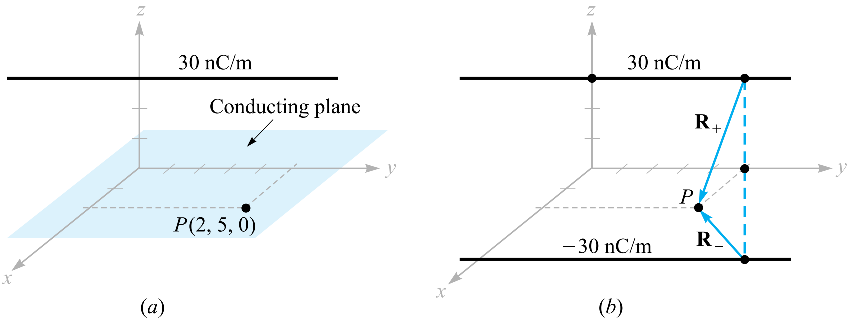

Figure 5.8 (*a*) A line charge above a conducting plane. (*b*) The conductor is removed, and the image of the line charge is added.

D5.6. A perfectly conducting plane is located in free space at $x = 4$, and a uniform infinite line charge of 40 nC/m lies along the line $x = 6$, $y = 3$. Let $V = 0$ at the conducting plane. At $P(7, -1, 5)$ find: (a) V; (b) $\mathbf{E}$.

Ans. 317 V; $-45.3\mathbf{a}_x - 99.2\mathbf{a}_y$ V/m

5.6 SEMICONDUCTORS

If we now turn our attention to an intrinsic semiconductor material, such as pure germanium or silicon, two types of current carriers are present, electrons and holes. The electrons are those from the top of the filled valence band that have received sufficient energy (usually thermal) to cross the relatively small forbidden band into the conduction band. The forbidden-band energy gap in typical semiconductors is of the order of one electronvolt. The vacancies left by these electrons represent unfilled energy states in the valence band which may also move from atom to atom in the crystal. The vacancy is called a hole, and many semiconductor properties may be described by treating the hole as if it had a positive charge of e, a mobility, μ_h, and an effective mass comparable to that of the electron. Both carriers move in an electric field, and they move in opposite directions; hence each contributes a component of the total current which is in the same direction as that provided by the other. The conductivity is therefore a function of both hole and electron concentrations and mobilities,

$$\boxed{\sigma = -\rho_e \mu_e + \rho_h \mu_h} \tag{19}$$

For pure, or *intrinsic,* silicon, the electron and hole mobilities are 0.12 and 0.025, respectively, whereas for germanium, the mobilities are, respectively, 0.36 and 0.17. These values are given in square meters per volt-second and range from 10 to 100 times as large as those for aluminum, copper, silver, and other metallic conductors.[6] These mobilities are given for a temperature of 300 K.

The electron and hole concentrations depend strongly on temperature. At 300 K the electron and hole volume charge densities are both 0.0024 C/m^{3}in magnitude in intrinsic silicon and 3.0 C/m^3 in intrinsic germanium. These values lead to conductivities of 0.000 35 S/m in silicon and 1.6 S/m in germanium. As temperature increases, the mobilities decrease, but the charge densities increase very rapidly. As a result, the conductivity of silicon increases by a factor of 10 as the temperature increases from 300 to about 330 K and decreases by a factor of 10 as the temperature drops from 300 to about 275 K. Note that the conductivity of the intrinsic semiconductor increases with temperature, whereas that of a metallic conductor decreases with temperature; this is one of the characteristic differences between the metallic conductors and the intrinsic semiconductors.

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[6] Mobility values for semiconductors are given in References 2, 3, and 5 listed at the end of this chapter.

Intrinsic semiconductors also satisfy the point form of Ohm's law; that is, the conductivity is reasonably constant with current density and with the direction of the current density.

The number of charge carriers and the conductivity may both be increased dramatically by adding very small amounts of impurities. *Donor* materials provide additional electrons and form *n-type* semiconductors, whereas *acceptors* furnish extra holes and form *p-type* materials. The process is known as *doping,* and a donor concentration in silicon as low as one part in 10^7 causes an increase in conductivity by a factor of 10^5.

The range of value of the conductivity is extreme as we go from the best insulating materials to semiconductors and the finest conductors. In siemens per meter, σ ranges from 10^{-17} for fused quartz, 10^{-7} for poor plastic insulators, and roughly unity for semiconductors to almost 10^8 for metallic conductors at room temperature. These values cover the remarkably large range of some 25 orders of magnitude.

D5.7. Using the values given in this section for the electron and hole mobilities in silicon at 300 K, and assuming hole and electron charge densities are 0.0029 C/m^3 and -0.0029 C/m^3, respectively, find: (*a*) the component of the conductivity due to holes; (*b*) the component of the conductivity due to electrons; (*c*) the conductivity.

Ans. $72.5\ \mu$S/m; $348\ \mu$S/m; $421\ \mu$S/m

WWW Quizzes

5.7 THE NATURE OF DIELECTRIC MATERIALS

A dielectric in an electric field can be viewed as a free-space arrangement of microscopic electric dipoles, each of which is composed of a positive and a negative charge whose centers do not quite coincide. These are not free charges, and they cannot contribute to the conduction process. Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called *bound* charges, in contrast to the free charges that determine conductivity. The bound charges can be treated as any other sources of the electrostatic field. Therefore, we would not need to introduce the dielectric constant as a new parameter or to deal with permittivities different from the permittivity of free space; however, the alternative would be to consider *every charge within a piece of dielectric material*. This is too great a price to pay for using all our previous equations in an unmodified form, and we shall therefore spend some time theorizing about dielectrics in a qualitative way; introducing polarization **P**, permittivity ϵ, and relative permittivity ϵ_r; and developing some quantitative relationships involving these new parameters.

WWW Interactives

The characteristic that all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their ability to store electric energy. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces.

This displacement against a restraining force is analogous to lifting a weight or stretching a spring and represents potential energy. The source of the energy is the external field, the motion of the shifting charges resulting perhaps in a transient current through a battery that is producing the field.

The actual mechanism of the charge displacement differs in the various dielectric materials. Some molecules, termed *polar* molecules, have a permanent displacement existing between the centers of "gravity" of the positive and negative charges, and each pair of charges acts as a dipole. Normally the dipoles are oriented in a random way throughout the interior of the material, and the action of the external field is to align these molecules, to some extent, in the same direction. A sufficiently strong field may even produce an additional displacement between the positive and negative charges.

A *nonpolar* molecule does not have this dipole arrangement until after a field is applied. The negative and positive charges shift in opposite directions against their mutual attraction and produce a dipole that is aligned with the electric field.

Either type of dipole may be described by its dipole moment **p**, as developed in Section 4.7, Eq. (36),

$$\mathbf{p} = Q\mathbf{d} \tag{20}$$

where Q is the positive one of the two bound charges composing the dipole, and **d** is the vector from the negative to the positive charge. We note again that the units of **p** are coulomb-meters.

If there are n dipoles per unit volume and we deal with a volume Δv, then there are $n\,\Delta v$ dipoles, and the total dipole moment is obtained by the vector sum,

$$\mathbf{p}_{\text{total}} = \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

If the dipoles are aligned in the same general direction, $\mathbf{p}_{\text{total}}$ may have a significant value. However, a random orientation may cause $\mathbf{p}_{\text{total}}$ to be essentially zero.

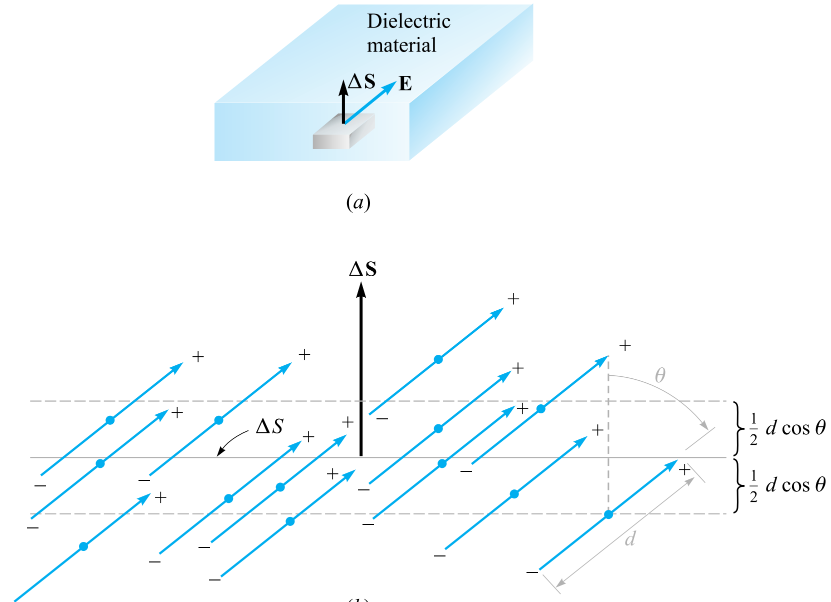
We now define the polarization **P** as the *dipole moment per unit volume,*

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i \tag{21}$$

with units of coulombs per square meter. We will treat **P** as a typical continuous field, even though it is obvious that it is essentially undefined at points within an atom or molecule. Instead, we should think of its value at any point as an average value taken over a sample volume Δv—large enough to contain many molecules ($n\,\Delta v$ in number), but yet sufficiently small to be considered incremental in concept.

Our immediate goal is to show that the bound-volume charge density acts like the free-volume charge density in producing an external field; we will obtain a result similar to Gauss's law.

To be specific, assume that we have a dielectric containing nonpolar molecules. No molecule has a dipole moment, and $\mathbf{P} = 0$ throughout the material. Somewhere in the interior of the dielectric we select an incremental surface element $\Delta\mathbf{S}$, as shown in Figure 5.9*a*, and apply an electric field **E**. The electric field produces a moment



(*a*)

(*b*)

Figure 5.9 (*a*) An incremental surface element Δ*S* is shown in the interior of a dielectric in which an electric field E is present. (*b*) The nonpolar molecules form dipole moments p and a polarization P. There is a net transfer of bound charge across Δ*S*.

$\mathbf{p} = Q\mathbf{d}$ in each molecule, such that $\mathbf{p}$ and $\mathbf{d}$ make an angle θ with $\Delta\mathbf{S}$, as indicated in Figure 5.9*b*.

The bound charges will now move across $\Delta\mathbf{S}$. Each of the charges associated with the creation of a dipole must have moved a distance $\frac{1}{2}d\cos\theta$ in the direction perpendicular to $\Delta\mathbf{S}$. Thus, any positive charges initially lying below the surface $\Delta\mathbf{S}$ and within the distance $\frac{1}{2}d\cos\theta$ of the surface must have crossed $\Delta\mathbf{S}$ going upward. Also, any negative charges initially lying above the surface and within that distance $(\frac{1}{2}d\cos\theta)$ from $\Delta\mathbf{S}$ must have crossed $\Delta\mathbf{S}$ going downward. Therefore, because there are n molecules/m^3, the net total charge that crosses the elemental surface in an upward direction is equal to $nQd\cos\theta\,\Delta S$, or

$$\Delta Q_b = nQ\mathbf{d}\cdot\Delta\mathbf{S}$$

where the subscript on Q_b reminds us that we are dealing with a bound charge and not a free charge. In terms of the polarization, we have

$$\Delta Q_b = \mathbf{P}\cdot\Delta\mathbf{S}$$

If we interpret $\Delta\mathbf{S}$ as an element of a *closed* surface inside the dielectric material, then the direction of $\Delta\mathbf{S}$ is outward, and the net increase in the bound charge *within* the closed surface is obtained through the integral

$$Q_b = -\oint_S \mathbf{P}\cdot d\mathbf{S} \tag{22}$$

This last relationship has some resemblance to Gauss's law, and we may now generalize our definition of electric flux density so that it applies to media other than free space. We first write Gauss's law in terms of $\epsilon_0\mathbf{E}$ and Q_T, the *total* enclosed charge, bound plus free:

$$Q_T = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \tag{23}$$

where

$$Q_T = Q_b + Q$$

and Q is the total *free* charge enclosed by the surface S. Note that the free charge appears without a subscript because it is the most important type of charge and will appear in Maxwell's equations.

Combining these last three equations, we obtain an expression for the free charge enclosed,

$$Q = Q_T - Q_b = \oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} \tag{24}$$

D is now defined in more general terms than was done in Chapter 3,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}} \tag{25}$$

There is thus an added term to **D** that appears when polarizable material is present. Thus,

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{26}$$

where Q is the free charge enclosed.

Utilizing the several volume charge densities, we have

$$Q_b = \int_v \rho_b \, dv$$

$$Q = \int_v \rho_v \, dv$$

$$Q_T = \int_v \rho_T \, dv$$

With the help of the divergence theorem, we may therefore transform Eqs. (22), (23), and (26) into the equivalent divergence relationships,

$$\nabla \cdot \mathbf{P} = -\rho_b$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_T$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \tag{27}$$

We will emphasize only Eq. (26) and (27), the two expressions involving the free charge, in the work that follows.

In order to make any real use of these new concepts, it is necessary to know the relationship between the electric field intensity **E** and the polarization **P** that results. This relationship will, of course, be a function of the type of material, and we will essentially limit our discussion to those isotropic materials for which **E** and **P** are linearly related. In an isotropic material, the vectors **E** and **P** are always parallel, regardless of the orientation of the field. Although most engineering dielectrics are linear for moderate-to-large field strengths and are also isotropic, single crystals may be anisotropic. The periodic nature of crystalline materials causes dipole moments to be formed most easily along the crystal axes, and not necessarily in the direction of the applied field.

In *ferroelectric* materials, the relationship between **P** and **E** not only is nonlinear, but also shows hysteresis effects; that is, the polarization produced by a given electric field intensity depends on the past history of the sample. Important examples of this type of dielectric are barium titanate, often used in ceramic capacitors, and Rochelle salt.

The linear relationship between **P** and **E** is

$$\boxed{\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}} \tag{28}$$

where χ_e (chi) is a dimensionless quantity called the *electric susceptibility* of the material.

Using this relationship in Eq. (25), we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$$

The expression within the parentheses is now defined as

$$\epsilon_r = \chi_e + 1 \tag{29}$$

This is another dimensionless quantity, and it is known as the *relative permittivity*, or *dielectric constant* of the material. Thus,

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \tag{30}$$

where

$$\boxed{\epsilon = \epsilon_0 \epsilon_r} \tag{31}$$

and ϵ is the *permittivity*. The dielectric constants are given for some representative materials in Appendix C.

Anisotropic dielectric materials cannot be described in terms of a simple susceptibility or permittivity parameter. Instead, we find that each component of **D** may be a function of every component of **E**, and $\mathbf{D} = \epsilon \mathbf{E}$ becomes a matrix equation where **D** and **E** are each 3 × 1 column matrices and ϵ is a 3 × 3 square matrix. Expanding the matrix equation gives

$$\begin{aligned} D_x &= \epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z \\ D_y &= \epsilon_{yx} E_x + \epsilon_{yy} E_y + \epsilon_{yz} E_z \\ D_z &= \epsilon_{zx} E_x + \epsilon_{zy} E_y + \epsilon_{zz} E_z \end{aligned}$$

Note that the elements of the matrix depend on the selection of the coordinate axes in the anisotropic material. Certain choices of axis directions lead to simpler matrices.[7]

D and **E** (and **P**) are no longer parallel, and although $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ remains a valid equation for anisotropic materials, we may continue to use $\mathbf{D} = \epsilon\mathbf{E}$ only by interpreting it as a matrix equation. We will concentrate our attention on linear isotropic materials and reserve the general case for a more advanced text.

In summary, then, we now have a relationship between **D** and **E** that depends on the dielectric material present,

$$\boxed{\mathbf{D} = \epsilon\mathbf{E}} \tag{30}$$

where

$$\boxed{\epsilon = \epsilon_0\epsilon_r} \tag{31}$$

This electric flux density is still related to the free charge by either the point or integral form of Gauss's law:

$$\boxed{\nabla\cdot\mathbf{D} = \rho_v} \tag{27}$$

$$\boxed{\oint_S \mathbf{D}\cdot d\mathbf{S} = Q} \tag{26}$$

Use of the relative permittivity, as indicated by Eq. (31), makes consideration of the polarization, dipole moments, and bound charge unnecessary. However, when anisotropic or nonlinear materials must be considered, the relative permittivity, in the simple scalar form that we have discussed, is no longer applicable.

EXAMPLE 5.4

We locate a slab of Teflon in the region $0 \le x \le a$, and assume free space where $x < 0$ and $x > a$. Outside the Teflon there is a uniform field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$ V/m. We seek values for **D**, **E**, and **P** everywhere.

Solution. The dielectric constant of the Teflon is 2.1, and thus the electric susceptibility is 1.1.

Outside the slab, we have immediately $\mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$. Also, as there is no dielectric material there, $\mathbf{P}_{\text{out}} = 0$. Now, any of the last four or five equations will enable us to relate the several fields inside the material to each other. Thus

$$\mathbf{D}_{\text{in}} = 2.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \le x \le a)$$

$$\mathbf{P}_{\text{in}} = 1.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \le x \le a)$$

---

[7] A more complete discussion of this matrix may be found in the Ramo, Whinnery, and Van Duzer reference listed at the end of this chapter.

As soon as we establish a value for any of these three fields within the dielectric, the other two can be found immediately. The difficulty lies in crossing over the boundary from the known fields external to the dielectric to the unknown ones within it. To do this we need a boundary condition, and this is the subject of the next section. We will complete this example then.

In the remainder of this text we will describe polarizable materials in terms of **D** and ϵ rather than **P** and χ_e. We will limit our discussion to isotropic materials.

D5.8. A slab of dielectric material has a relative dielectric constant of 3.8 and contains a uniform electric flux density of 8 nC/m^2. If the material is lossless, find: (*a*) E; (*b*) P; (*c*) the average number of dipoles per cubic meter if the average dipole moment is $10^{-29}\,\text{C}\cdot\text{m}$.

Ans. 238 V/m; 5.89 nC/m^2; $5.89 \times 10^{20}\ \text{m}^{-3}$

5.8 BOUNDARY CONDITIONS FOR PERFECT DIELECTRIC MATERIALS

WW Illustrations

How do we attack a problem in which there are two different dielectrics, or a dielectric and a conductor? This is another example of a *boundary condition*, such as the condition at the surface of a conductor whereby the tangential fields are zero and the normal electric flux density is equal to the surface charge density on the conductor. Now we take the first step in solving a two-dielectric problem, or a dielectric-conductor problem, by determining the behavior of the fields at the dielectric interface.

Let us first consider the interface between two dielectrics having permittivities ϵ_1 and ϵ_2 and occupying regions 1 and 2, as shown in Figure 5.10. We first examine

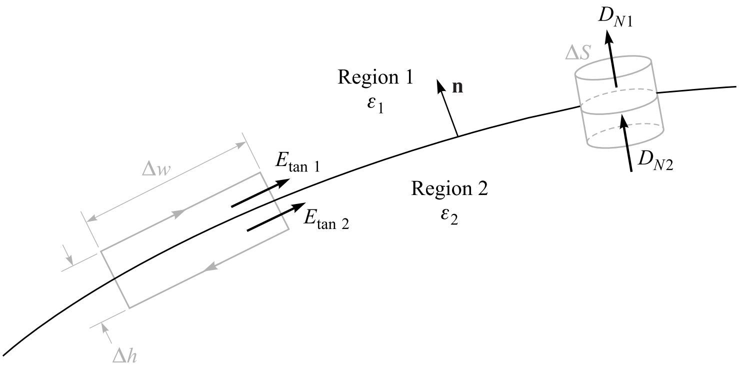


Figure 5.10 The boundary between perfect dielectrics of permittivities ϵ_1 and ϵ_2. The continuity of D_N is shown by the gaussian surface on the right, and the continuity of $E_{\tan}$ is shown by the line integral about the closed path at the left.

the tangential components by using

$$\oint \mathbf{E}\cdot d\mathbf{L} = 0$$

around the small closed path on the left, obtaining

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

The small contribution to the line integral by the normal component of **E** along the sections of length Δh becomes negligible as Δh decreases and the closed path crowds the surface. Immediately, then,

$$\boxed{E_{\tan 1} = E_{\tan 2}} \tag{32}$$

Evidently, Kirchhoff's voltage law is still applicable to this case. Certainly we have shown that the potential difference between any two points on the boundary that are separated by a distance Δw is the same immediately above or below the boundary.

If the tangential electric field intensity is continuous across the boundary, then tangential **D** is discontinuous, for

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

or

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \tag{33}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in Figure 5.10. The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

from which

$$\boxed{D_{N1} - D_{N2} = \rho_S} \tag{34}$$

What is this surface charge density? It cannot be a *bound* surface charge density, because we are taking the polarization of the dielectric into effect by using a dielectric constant different from unity; that is, instead of considering bound charges in free space, we are using an increased permittivity. Also, it is extremely unlikely that any *free* charge is on the interface, for no free charge is available in the perfect dielectrics we are considering. This charge must then have been placed there deliberately, thus unbalancing the total charge in and on this dielectric body. Except for this special case, then, we may assume ρ_S is zero on the interface and

$$\boxed{D_{N1} = D_{N2}} \tag{35}$$

or the normal component of **D** is continuous. It follows that

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2} \tag{36}$$

and normal **E** is discontinuous.

Equations (32) and (34) can be written in terms of field vectors in any direction, along with the unit normal to the surface as shown in Figure 5.10. Formally stated, the boundary conditions for the electric flux density and the electric field strength at the surface of a perfect dielectric are

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{n} = \rho_s \tag{37}$$

which is the general statement of Eq. (32), and

$$(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{n} = 0 \tag{38}$$

generally states Eq. (34). This construction was used previously in Eqs. (17) and (18) for a conducting surface, in which the normal or tangential components of the fields are obtained through the dot product or cross product with the normal, respectively.

These conditions may be used to show the change in the vectors **D** and **E** at the surface. Let $\mathbf{D}_1$ (and $\mathbf{E}_1$) make an angle θ_1 with a normal to the surface (Figure 5.11). Because the normal components of **D** are continuous,

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2} \tag{39}$$

The ratio of the tangential components is given by (33) as

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

or

$$\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2 \tag{40}$$

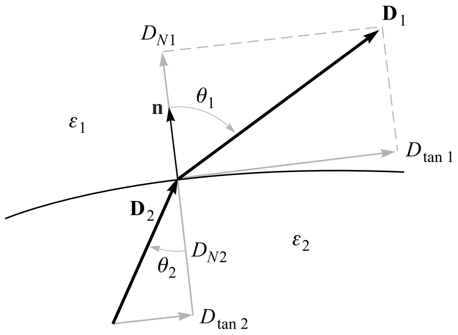


Figure 5.11 The refraction of **D** at a dielectric interface. For the case shown, $\epsilon_1 > \epsilon_2$; $\mathbf{E}_1$ and $\mathbf{E}_2$ are directed along $\mathbf{D}_1$ and $\mathbf{D}_2$, with $D_1 > D_2$ and $E_1 < E_2$.

and the division of this equation by (39) gives

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2} \tag{41}$$

In Figure 5.11 we have assumed that $\epsilon_1 > \epsilon_2$, and therefore $\theta_1 > \theta_2$.

The direction of $\mathbf{E}$ on each side of the boundary is identical with the direction of $\mathbf{D}$, because $\mathbf{D} = \epsilon\mathbf{E}$.

The magnitude of $\mathbf{D}$ in region 2 may be found from Eq. (39) and (40),

$$D_2 = D_1\sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1} \tag{42}$$

and the magnitude of $\mathbf{E}_2$ is

$$E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1} \tag{43}$$

An inspection of these equations shows that D is larger in the region of larger permittivity (unless $\theta_1 = \theta_2 = 0^\circ$ where the magnitude is unchanged) and that E is larger in the region of smaller permittivity (unless $\theta_1 = \theta_2 = 90^\circ$, where its magnitude is unchanged).

EXAMPLE 5.5

Complete Example 5.4 by finding the fields within the Teflon ($\epsilon_r = 2.1$), given the uniform external field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$ in free space.

Solution. We recall that we had a slab of Teflon extending from $x = 0$ to $x = a$, as shown in Figure 5.12, with free space on both sides of it and an external field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$. We also have $\mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$ and $\mathbf{P}_{\text{out}} = 0$.

Inside, the continuity of D_N at the boundary allows us to find that $\mathbf{D}_{\text{in}} = \mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$. This gives us $\mathbf{E}_{\text{in}} = \mathbf{D}_{\text{in}}/\epsilon = \epsilon_0 E_0\mathbf{a}_x/(\epsilon_r\epsilon_0) = 0.476E_0\mathbf{a}_x$. To get the polarization field in the dielectric, we use $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and obtain

$$\mathbf{P}_{\text{in}} = \mathbf{D}_{\text{in}} - \epsilon_0\mathbf{E}_{\text{in}} = \epsilon_0 E_0\mathbf{a}_x - 0.476\epsilon_0 E_0\mathbf{a}_x = 0.524\epsilon_0 E_0\mathbf{a}_x$$

Summarizing then gives

$$\begin{aligned}
\mathbf{D}_{\text{in}} &= \epsilon_0 E_0\mathbf{a}_x & &(0 \le x \le a)\\
\mathbf{E}_{\text{in}} &= 0.476E_0\mathbf{a}_x & &(0 \le x \le a)\\
\mathbf{P}_{\text{in}} &= 0.524\epsilon_0 E_0\mathbf{a}_x & &(0 \le x \le a)
\end{aligned}$$

A practical problem most often does not provide us with a direct knowledge of the field on either side of the boundary. The boundary conditions must be used to help us determine the fields on both sides of the boundary from the other information that is given.

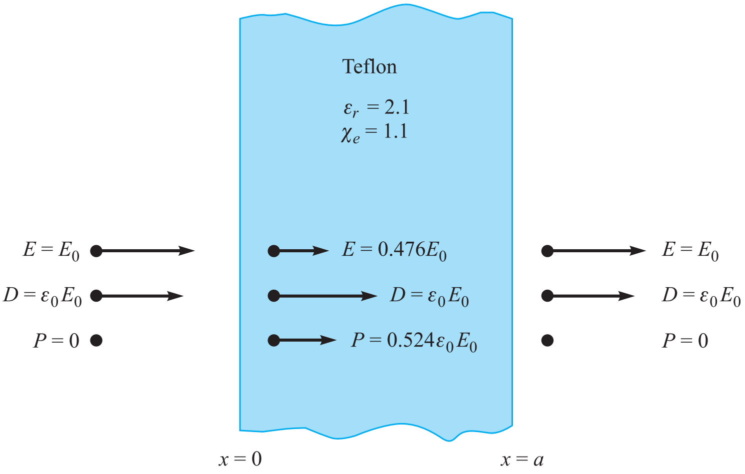


Figure 5.12 A knowledge of the electric field external to the dielectric enables us to find the remaining external fields first and then to use the continuity of normal D to begin finding the internal fields.

D5.9. Let Region 1 ($z < 0$) be composed of a uniform dielectric material for which $\epsilon_r = 3.2$, while Region 2 ($z > 0$) is characterized by $\epsilon_r = 2$. Let $\mathbf{D}_1 = -30\mathbf{a}_x + 50\mathbf{a}_y + 70\mathbf{a}_z$ nC/m^2 and find: (*a*) D_{N1}; (*b*) $\mathbf{D}_{t1}$; (*c*) D_{t1}; (*d*) D_1; (*e*) θ_1; (*f*) $\mathbf{P}_1$.

Ans. 70 nC/m^2; $-30\mathbf{a}_x + 50\mathbf{a}_y$ nC/m^2; 58.3 nC/m^2; 91.1 nC/m^2; 39.8°; $-20.6\mathbf{a}_x + 34.4\mathbf{a}_y + 48.1\mathbf{a}_z$ nC/m^2

D5.10. Continue Problem D5.9 by finding: (*a*) $\mathbf{D}_{N2}$; (*b*) $\mathbf{D}_{t2}$; (*c*) $\mathbf{D}_2$; (*d*) $\mathbf{P}_2$; (*e*) θ_2.

Ans. $70\mathbf{a}_z$ nC/m^2; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y$ nC/m^2; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y + 70\mathbf{a}_z$ nC/m^2; $-9.38\mathbf{a}_x + 15.63\mathbf{a}_y + 35\mathbf{a}_z$ nC/m^2; 27.5°

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CHAPTER 5 PROBLEMS

5.1 Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]$ kA/m^2 (*a*) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1, 0 < z < 2$. (*b*) Find the total current leaving the region $0 < x, y < 1, 2 < z < 3$ by integrating $\mathbf{J} \cdot d\mathbf{S}$ over the surface of the cube. (*c*) Repeat part (*b*), but use the divergence theorem.

5.2 Given $\mathbf{J} = -10^{-4}(y\mathbf{a}_x + x\mathbf{a}_y)$ A/m^2, find the current crossing the $y = 0$ plane in the $-\mathbf{a}_y$ direction between $z = 0$ and 1, and $x = 0$ and 2.

5.3 Let $\mathbf{J} = 400 \sin\theta/(r^2 + 4)\, \mathbf{a}_r$ A/m^2. (*a*) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi, 0 < \phi < 2\pi$. (*b*) Find the average value of $\mathbf{J}$ over the defined area.

5.4 If volume charge density is given as $\rho_v = (\cos\omega t)/r^2$ C/m^2 in spherical coordinates, find $\mathbf{J}$. It is reasonable to assume that $\mathbf{J}$ is not a function of θ or ϕ.

5.5 Let $\mathbf{J} = 25/\rho\mathbf{a}_\rho - 20/(\rho^2 + 0.01)\, \mathbf{a}_z$ A/m^2. (*a*) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$. (*b*) Calculate $\partial\rho_v/\partial t$. (*c*) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$. (*d*) Show that the divergence theorem is satisified for $\mathbf{J}$ and the surface specified in part (*c*).

5.6 In spherical coordinates, a current density $\mathbf{J} = -k/(r \sin\theta)\, \mathbf{a}_\theta$ A/m^2 exists in a conducting medium, where k is a constant. Determine the total current in the $\mathbf{a}_z$ direction that crosses a circular disk of radius R, centered on the z axis and located at (*a*) $z = 0$; (*b*) $z = h$.

5.7 Assuming that there is no transformation of mass to energy or vice versa, it is possible to write a continuity equation for mass. (*a*) If we use the continuity equation for charge as our model, what quantities correspond to $\mathbf{J}$ and ρ_v? (*b*) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, −9.85, 1.75, −2.00, −4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center.

5.8 A truncated cone has a height of 16 cm. The circular faces on the top and bottom have radii of 2 mm and 0.1 mm, respectively. If the material from which this solid cone is constructed has a conductivity of 2×10^6 S/m, use some good approximations to determine the resistance between the two circular faces.

5.9 (*a*) Using data tabulated in Appendix C, calculate the required diameter for a 2-m-long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it. (*b*) Calculate the rms current density in the wire.

5.10 A large brass washer has a 2-cm inside diameter, a 5-cm outside diameter, and is 0.5 cm thick. Its conductivity is $\sigma = 1.5 \times 10^7$ S/m. The washer is cut in half along a diameter, and a voltage is applied between the two rectangular faces of one part. The resultant electric field in the interior of the half-washer is $\mathbf{E} = (0.5/\rho)\,\mathbf{a}_\phi$ V/m in cylindrical coordinates, where the z axis is the axis of the washer. (*a*) What potential difference exists between the two rectangular faces? (*b*) What total current is flowing? (*c*) What is the resistance between the two faces?

5.11 Two perfectly conducting cylindrical surfaces of length ℓ are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc. (*a*) Find the voltage and resistance between the cylinders, and **E** in the region between the cylinders, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < \rho < 5$ cm. (*b*) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power.

5.12 Two identical conducting plates, each having area A, are located at $z = 0$ and $z = d$. The region between plates is filled with a material having z-dependent conductivity, $\sigma(z) = \sigma_0 e^{-z/d}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters, (*a*) the resistance of the material; (*b*) the total current flowing between plates; (*c*) the electric field intensity **E** within the material.

5.13 A hollow cylindrical tube with a rectangular cross section has external dimensions of 0.5 in. by 1 in. and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5 \times 10^7$ S/m. A current of 200 A dc is flowing down the tube. (*a*) What voltage drop is present across a 1 m length of the tube? (*b*) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5 \times 10^5$ S/m.

5.14 A rectangular conducting plate lies in the xy plane, occupying the region $0 < x < a, 0 < y < b$. An identical conducting plate is positioned directly above and parallel to the first, at $z = d$. The region between plates is filled with material having conductivity $\sigma(x) = \sigma_0 e^{-x/a}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters, (*a*) the electric field intensity **E** within the material; (*b*) the total current flowing between plates; (*c*) the resistance of the material.

5.15 Let $V = 10(\rho + 1)z^2 \cos\phi$ V in free space. (a) Let the equipotential surface $V = 20$ V define a conductor surface. Find the equation of the conductor surface. (b) Find ρ and **E** at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$. (c) Find $|\rho_S|$ at that point.

5.16 A coaxial transmission line has inner and outer conductor radii a and b. Between conductors ($a < \rho < b$) lies a conductive medium whose conductivity is $\sigma(\rho) = \sigma_0/\rho$, where σ_0 is a constant. The inner conductor is charged to potential V_0, and the outer conductor is grounded. (a) Assuming dc radial current I per unit length in z, determine the radial current density field **J** in A/m^2. (b) Determine the electric field intensity **E** in terms of I and other parameters, given or known. (c) By taking an appropriate line integral of **E** as found in part (b), find an expression that relates V_0 to I. (d) Find an expression for the conductance of the line per unit length, G.

5.17 Given the potential field $V = 100xz/(x^2 + 4)$ V in free space: (a) Find **D** at the surface $z = 0$. (b) Show that the $z = 0$ surface is an equipotential surface. (c) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$.

5.18 Two parallel circular plates of radius a are located at $z = 0$ and $z = d$. The top plate ($z = d$) is raised to potential V_0; the bottom plate is grounded. Between the plates is a conducting material having radial-dependent conductivity, $\sigma(\rho) = \sigma_0\rho$, where σ_0 is a constant. (a) Find the ρ-*independent* electric field strength, **E**, between plates. (b) Find the current density, **J** between plates. (c) Find the total current, I, in the structure. (d) Find the resistance between plates.

5.19 Let $V = 20x^2yz - 10z^2$ V in free space. (a) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V. (b) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \le V \le 60$ V is the field-containing region. (c) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface.

5.20 Two point charges of $-100\pi\,\mu$C are located at $(2, -1, 0)$ and $(2, 1, 0)$. The surface $x = 0$ is a conducting plane. (a) Determine the surface charge density at the origin. (b) Determine ρ_S at $P(0, h, 0)$.

5.21 Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$. (a) Let $V = 0$ at the plane $y = 0$, and find V at $P(1, 2, 0)$. (b) Find **E** at P.

5.22 The line segment $x = 0$, $-1 \le y \le 1$, $z = 1$, carries a linear charge density $\rho_L = \pi|y|\,\mu$C/m. Let $z = 0$ be a conducting plane and determine the surface charge density at: (a) $(0, 0, 0)$; (b) $(0, 1, 0)$.

5.23 A dipole with $\mathbf{p} = 0.1\mathbf{a}_z\ \mu\text{C}\cdot\text{m}$ is located at $A(1, 0, 0)$ in free space, and the $x = 0$ plane is perfectly conducting. (*a*) Find V at $P(2, 0, 1)$. (*b*) Find the equation of the 200 V equipotential surface in rectangular coordinates.

5.24 At a certain temperature, the electron and hole mobilities in intrinsic germanium are given as 0.43 and 0.21 $\text{m}^2/\text{V}\cdot\text{s}$, respectively. If the electron and hole concentrations are both $2.3 \times 10^{19}\ \text{m}^{-3}$, find the conductivity at this temperature.

5.25 Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}\ \text{C/m}^3$. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}\ \text{m}^2/\text{V}\cdot\text{s}$ and $\mu_e = 2.1 \times 10^5 T^{-2.5}\ \text{m}^2/\text{V}\cdot\text{s}$, where the temperature, T, is in degrees Kelvin. Find σ at: (*a*) 0°C; (*b*) 40°C; (*c*) 80°C.

5.26 A semiconductor sample has a rectangular cross section 1.5 by 2.0 mm, and a length of 11.0 mm. The material has electron and hole densities of 1.8×10^{18} and $3.0 \times 10^{15}\ \text{m}^{-3}$, respectively. If $\mu_e = 0.082\ \text{m}^2/\text{V}\cdot\text{s}$ and $\mu_h = 0.0021\ \text{m}^2/\ \text{V}\cdot\text{s}$, find the resistance offered between the end faces of the sample.

5.27 Atomic hydrogen contains 5.5×10^{25} atoms/m^{3}at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m. (*a*) Find P. (*b*) Find ϵ_r.

5.28 Find the dielectric constant of a material in which the electric flux density is four times the polarization.

5.29 A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_r = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho\ \text{nC/m}^2$ in the dielectric, find (*a*) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ; (*b*) V_{ab} and χ_e. (*c*) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$.

5.30 Consider a composite material made up of two species, having number densities N_1 and N_2 molecules/m^3, respectively. The two materials are uniformly mixed, yielding a total number density of $N = N_1 + N_2$. The presence of an electric field $\mathbf{E}$ induces molecular dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ within the individual species, whether mixed or not. Show that the dielectric constant of the composite material is given by $\epsilon_r = f\epsilon_{r1} + (1 - f)\epsilon_{r2}$, where f is the number fraction of species 1 dipoles in the composite, and where ϵ_{r1} and ϵ_{r2} are the dielectric constants that the unmixed species would have if each had number density N.

5.31 The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_r = \epsilon_{r1} = 3$, while $\epsilon_{r2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find (*a*) E_{N1}; (*b*) $\mathbf{E}_{T1}$; (*c*) $\mathbf{E}_1$; (*d*) the angle θ_1 between $\mathbf{E}_1$ and a normal to the surface; (*e*) D_{N2}; (*f*) D_{T2}; (*g*) $\mathbf{D}_2$; (*h*) $\mathbf{P}_2$; (*i*) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface.

5.32 Two equal but opposite-sign point charges of 3 μC are held x meters apart by a spring that provides a repulsive force given by $F_{sp} = 12(0.5 - x)$ N. Without any force of attraction, the spring would be fully extended to 0.5 m. (*a*) Determine the charge separation. (*b*) What is the dipole moment?

5.33 Two perfect dielectrics have relative permittivities $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$.

5.34 Region 1 ($x \geq 0$) is a dielectric with $\epsilon_{r1} = 2$, while region 2($x < 0$) has $\epsilon_{r2} = 5$. Let $\mathbf{E}_1 = 20\mathbf{a}_x - 10\mathbf{a}_y + 50\mathbf{a}_z$ V/m. (*a*) Find $\mathbf{D}_2$. (*b*) Find the energy density in both regions.

5.35 Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{r1} = 2$ for $0 < \phi < \pi/2$ and $\epsilon_{r2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find (*a*) $\mathbf{E}_2$; (*b*) the total electrostatic energy stored in a 1 m length of each region.

CHAPTER 6

Capacitance

Capacitance measures the capability of energy storage in electrical devices. It can be deliberately designed for a specific purpose, or it may exist as an unavoidable by-product of the device structure that one must live with. Understanding capacitance and its impact on device or system operation is critical in every aspect of electrical engineering.

A capacitor is a device that stores energy; energy thus stored can either be associated with accumulated charge or it can be related to the stored electric field, as was discussed in Section 4.8. In fact, one can think of a capacitor as a device that stores electric *flux*, in a similar way that an inductor — an analogous device — stores magnetic flux (or ultimately magnetic field energy). We will explore this in Chapter 8. A primary goal in this chapter is to present the methods for calculating capacitance for a number of cases, including transmission line geometries, and to be able to make judgments on how capacitance will be altered by changes in materials or their configuration. ■

6.1 CAPACITANCE DEFINED

Consider two conductors embedded in a homogeneous dielectric (Figure 6.1). Conductor M_2 carries a total positive charge Q, and M_1 carries an equal negative charge. There are no other charges present, and the *total* charge of the system is zero.

We now know that the charge is carried on the surface as a surface charge density and also that the electric field is normal to the conductor surface. Each conductor is, moreover, an equipotential surface. Because M_2 carries the positive charge, the electric flux is directed from M_2 to M_1, and M_2 is at the more positive potential. In other words, work must be done to carry a positive charge from M_1 to M_2.

Let us designate the potential difference between M_2 and M_1 as V_0. We may now define the *capacitance* of this two-conductor system as the ratio of the magnitude

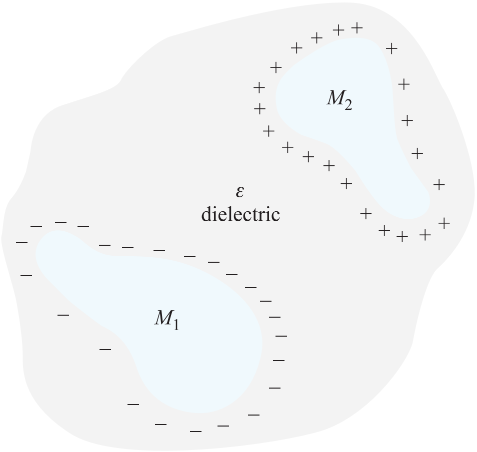

Figure 6.1 Two oppositely charged conductors M_1 and M_2 surrounded by a uniform dielectric. The ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them is the capacitance C.

of the total charge on either conductor to the magnitude of the potential difference between conductors,

$$\boxed{C = \frac{Q}{V_0}} \tag{1}$$

In general terms, we determine Q by a surface integral over the positive conductors, and we find V_0 by carrying a unit positive charge from the negative to the positive surface,

$$C = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot d\mathbf{L}} \tag{2}$$

The capacitance is independent of the potential and total charge, for their ratio is constant. If the charge density is increased by a factor of N, Gauss's law indicates that the electric flux density or electric field intensity also increases by N, as does the potential difference. The capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the homogeneous dielectric.

Capacitance is measured in *farads* (F), where a farad is defined as one coulomb per volt. Common values of capacitance are apt to be very small fractions of a farad, and consequently more practical units are the microfarad (μF), the nanofarad (nF), and the picofarad (pF).

6.2 PARALLEL-PLATE CAPACITOR

We can apply the definition of capacitance to a simple two-conductor system in which the conductors are identical, infinite parallel planes with separation d (Figure 6.2). Choosing the lower conducting plane at $z = 0$ and the upper one at $z = d$, a uniform sheet of surface charge $\pm\rho_S$ on each conductor leads to the uniform field [Section 2.5, Eq. (18)]

$$\mathbf{E} = \frac{\rho_S}{\epsilon}\mathbf{a}_z$$

where the permittivity of the homogeneous dielectric is ϵ, and

$$\mathbf{D} = \rho_S \mathbf{a}_z$$

Note that this result could be obtained by applying the boundary condition at a conducting surface (Eq. (18), Chapter 5) at either *one* of the plate surfaces. Referring to the surfaces and their unit normal vectors in Fig. 6.2, where $\mathbf{n}_\ell = \mathbf{a}_z$ and $\mathbf{n}_u = -\mathbf{a}_z$, we find on the lower plane:

$$\mathbf{D}\cdot\mathbf{n}_\ell\Big|_{z=0} = \mathbf{D}\cdot\mathbf{a}_z = \rho_s \;\Rightarrow\; \mathbf{D} = \rho_s\,\mathbf{a}_z$$

On the upper plane, we get the same result

$$\mathbf{D}\cdot\mathbf{n}_u\Big|_{z=d} = \mathbf{D}\cdot(-\mathbf{a}_z) = -\rho_s \;\Rightarrow\; \mathbf{D} = \rho_s\,\mathbf{a}_z$$

This is a key advantage of the conductor boundary condition, in that we need to apply it only to a single boundary to obtain the *total* field there (arising from all other sources).

The potential difference between lower and upper planes is

$$V_0 = -\int_{\text{upper}}^{\text{lower}} \mathbf{E}\cdot d\mathbf{L} = -\int_d^0 \frac{\rho_S}{\epsilon}dz = \frac{\rho_S}{\epsilon}d$$

Since the total charge on either plane is infinite, the capacitance is infinite. A more practical answer is obtained by considering planes, each of area S, whose linear dimensions are much greater than their separation d. The electric field and charge

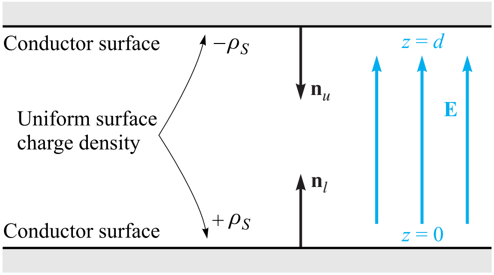

Figure 6.2 The problem of the parallel-plate capacitor. The capacitance per square meter of surface area is ϵ/d.

distribution are then almost uniform at all points not adjacent to the edges, and this latter region contributes only a small percentage of the total capacitance, allowing us to write the familiar result

$$Q = \rho_S S$$

$$V_0 = \frac{\rho_S}{\epsilon} d$$

$$\boxed{C = \frac{Q}{V_0} = \frac{\epsilon S}{d}} \qquad (3)$$

More rigorously, we might consider Eq. (3) as the capacitance of a portion of the infinite-plane arrangement having a surface area S. Methods of calculating the effect of the unknown and nonuniform distribution near the edges must wait until we are able to solve more complicated potential problems.

EXAMPLE 6.1

Calculate the capacitance of a parallel-plate capacitor having a mica dielectric, $\epsilon_r = 6$, a plate area of 10 in.^{2}, and a separation of 0.01 in.

Solution. We may find that

$$S = 10 \times 0.0254^2 = 6.45 \times 10^{-3}\ \text{m}^2$$

$$d = 0.01 \times 0.0254 = 2.54 \times 10^{-4}\ \text{m}$$

and therefore

$$C = \frac{6 \times 8.854 \times 10^{-12} \times 6.45 \times 10^{-3}}{2.54 \times 10^{-4}} = 1.349\ \text{nF}$$

A large plate area is obtained in capacitors of small physical dimensions by stacking smaller plates in 50- or 100-decker sandwiches, or by rolling up foil plates separated by a flexible dielectric.

Table C.1 in Appendix C also indicates that materials are available having dielectric constants greater than 1000.

Finally, the total energy stored in the capacitor is

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \epsilon E^2\, dv = \tfrac{1}{2}\int_0^S \int_0^d \frac{\epsilon \rho_S^2}{\epsilon^2} dz\, dS = \tfrac{1}{2}\frac{\rho_S^2}{\epsilon} Sd = \tfrac{1}{2}\frac{\epsilon S}{d}\frac{\rho_S^2 d^2}{\epsilon^2}$$

or

$$\boxed{W_E = \tfrac{1}{2} C V_0^2 = \tfrac{1}{2} Q V_0 = \tfrac{1}{2}\frac{Q^2}{C}} \qquad (4)$$

which are all familiar expressions. Equation (4) also indicates that the energy stored in a capacitor with a fixed potential difference across it increases as the dielectric constant of the medium increases.

D6.1. Find the relative permittivity of the dielectric material present in a parallel-plate capacitor if: (*a*) $S = 0.12\,\text{m}^2$, $d = 80\,\mu\text{m}$, $V_0 = 12$ V, and the capacitor contains $1\,\mu$J of energy; (*b*) the stored energy density is 100 J/m^3, $V_0 = 200$ V, and $d = 45\,\mu$m; (*c*) $E = 200$ kV/m and $\rho_S = 20\,\mu\text{C/m}^2$.

Ans. 1.05; 1.14; 11.3

6.3 SEVERAL CAPACITANCE EXAMPLES

As a first brief example, we choose a coaxial cable or coaxial capacitor of inner radius a, outer radius b, and length L. No great derivational struggle is required, because the potential difference is given as Eq. (11) in Section 4.3, and we find the capacitance very simply by dividing this by the total charge $\rho_L L$ in the length L. Thus,

$$\boxed{C = \frac{2\pi\epsilon L}{\ln(b/a)}} \tag{5}$$

Next we consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b, $b > a$. The expression for the electric field was obtained previously by Gauss's law,

$$E_r = \frac{Q}{4\pi\epsilon r^2}$$

where the region between the spheres is a dielectric with permittivity ϵ. The expression for potential difference was found from this by the line integral [Section 4.3, Eq. (12)]. Thus,

$$V_{ab} = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Here Q represents the total charge on the inner sphere, and the capacitance becomes

$$\boxed{C = \frac{Q}{V_{ab}} = \frac{4\pi\epsilon}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{6}$$

If we allow the outer sphere to become infinitely large, we obtain the capacitance of an isolated spherical conductor,

$$\boxed{C = 4\pi\epsilon a} \tag{7}$$

For a diameter of 1 cm, or a sphere about the size of a marble,

$$C = 0.556\text{ pF}$$

in free space.

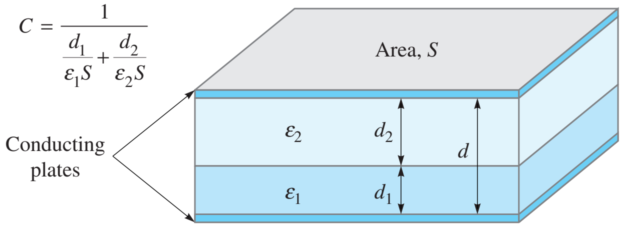

Figure 6.3 A parallel-plate capacitor containing two dielectrics with the dielectric interface parallel to the conducting plates.

Coating this sphere with a different dielectric layer, for which $\epsilon = \epsilon_1$, extending from $r = a$ to $r = r_1$,

$$D_r = \frac{Q}{4\pi r^2}$$

$$E_r = \frac{Q}{4\pi\epsilon_1 r^2} \qquad (a < r < r_1)$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r_1 < r)$$

and the potential difference is

$$V_a - V_\infty = -\int_{r_1}^{a} \frac{Q\,dr}{4\pi\epsilon_1 r^2} - \int_{\infty}^{r_1} \frac{Q\,dr}{4\pi\epsilon_0 r^2}$$

$$= \frac{Q}{4\pi}\left[\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}\right]$$

Therefore,

$$C = \frac{4\pi}{\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}} \tag{8}$$

In order to look at the problem of multiple dielectrics a little more thoroughly, let us consider a parallel-plate capacitor of area S and spacing d, with the usual assumption that d is small compared to the linear dimensions of the plates. The capacitance is $\epsilon_1 S/d$, using a dielectric of permittivity ϵ_1. Now replace a part of this dielectric by another of permittivity ϵ_2, placing the boundary between the two dielectrics parallel to the plates (Figure 6.3).

Some of us may immediately suspect that this combination is effectively two capacitors in series, yielding a total capacitance of

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

where $C_1 = \epsilon_1 S/d_1$ and $C_2 = \epsilon_2 S/d_2$. This is the correct result, but we can obtain it using less intuition and a more basic approach.

Because the capacitance definition, $C = Q/V$, involves a charge and a voltage, we may assume either and then find the other in terms of it. The capacitance is not a function of either, but only of the dielectrics and the geometry. Suppose we assume a potential difference V_0 between the plates. The electric field intensities in the two regions, E_2 and E_1, are both uniform, and $V_0 = E_1 d_1 + E_2 d_2$. At the dielectric interface, E is normal, and our boundary condition, Eq. (35) Chapter 5, tells us that $D_{N1} = D_{N2}$, or $\epsilon_1 E_1 = \epsilon_2 E_2$. This assumes (correctly) that there is no surface charge at the interface. Eliminating E_2 in our V_0 relation, we have

$$E_1 = \frac{V_0}{d_1 + d_2(\epsilon_1/\epsilon_2)}$$

and the surface charge density on the lower plate therefore has the magnitude

$$\rho_{S1} = D_1 = \epsilon_1 E_1 = \frac{V_0}{\dfrac{d_1}{\epsilon_1} + \dfrac{d_2}{\epsilon_2}}$$

Because $D_1 = D_2$, the magnitude of the surface charge is the same on each plate. The capacitance is then

$$C = \frac{Q}{V_0} = \frac{\rho_S S}{V_0} = \frac{1}{\dfrac{d_1}{\epsilon_1 S} + \dfrac{d_2}{\epsilon_2 S}} = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$$

As an alternate (and slightly simpler) solution, we might assume a charge Q on one plate, leading to a charge density Q/S and a value of D that is also Q/S. This is true in both regions, as $D_{N1} = D_{N2}$ and D is normal. Then $E_1 = D/\epsilon_1 = Q/(\epsilon_1 S)$, $E_2 = D/\epsilon_2 = Q/(\epsilon_2 S)$, and the potential differences across the regions are $V_1 = E_1 d_1 = Qd_1/(\epsilon_1 S)$, and $V_2 = E_2 d_2 = Qd_2/(\epsilon_2 S)$. The capacitance is

$$C = \frac{Q}{V} = \frac{Q}{V_1 + V_2} = \frac{1}{\dfrac{d_1}{\epsilon_1 S} + \dfrac{d_2}{\epsilon_2 S}} \tag{9}$$

How would the method of solution or the answer change if there were a third conducting plane along the interface? We would now expect to find surface charge on each side of this conductor, and the magnitudes of these charges should be equal. In other words, we think of the electric lines not as passing directly from one outer plate to the other, but as terminating on one side of this interior plane and then continuing on the other side. The capacitance is unchanged, provided, of course, that the added conductor is of negligible thickness. The addition of a thick conducting plate will increase the capacitance if the separation of the outer plates is kept constant, and this is an example of a more general theorem which states that the replacement of any portion of the dielectric by a conducting body will cause an increase in the capacitance.

If the dielectric boundary were placed *normal* to the two conducting plates and the dielectrics occupied areas of S_1 and S_2, then an assumed potential difference V_0 would produce field strengths $E_1 = E_2 = V_0/d$. These are tangential fields at the