**1.25** Given point $P(r = 0.8, \theta = 30^\circ, \phi = 45^\circ)$ and $\mathbf{E} = 1/r^2\ [\cos\phi\, \mathbf{a}_r + (\sin\phi/\sin\theta)\, \mathbf{a}_\phi]$, find (*a*) $\mathbf{E}$ at $P$; (*b*) $|\mathbf{E}|$ at $P$; (*c*) a unit vector in the direction of $\mathbf{E}$ at $P$.

**1.26** Express the uniform vector field $\mathbf{F} = 5\mathbf{a}_x$ in (*a*) cylindrical components; (*b*) spherical components.

**1.27** The surfaces $r = 2$ and 4, $\theta = 30^\circ$ and $50^\circ$, and $\phi = 20^\circ$ and $60^\circ$ identify a closed surface. Find (*a*) the enclosed volume; (*b*) the total area of the enclosing surface; (*c*) the total length of the twelve edges of the surface; (*d*) the length of the longest straight line that lies entirely within the surface.

**1.28** State whether or not $\mathbf{A} = \mathbf{B}$ and, if not, what conditions are imposed on $\mathbf{A}$ and $\mathbf{B}$ when (*a*) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$; (*b*) $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$; (*c*) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$ *and* $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$; (*d*) $\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$ *and* $\mathbf{A} \times \mathbf{C} = \mathbf{B} \times \mathbf{C}$ where $\mathbf{C}$ is any vector except $\mathbf{C} = 0$.

**1.29** Express the unit vector $\mathbf{a}_x$ in spherical components at the point: (*a*) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad; (*b*) $x = 3$, $y = 2$, $z = -1$; (*c*) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$.

**1.30** Consider a problem analogous to the varying wind velocities encountered by transcontinental aircraft. We assume a constant altitude, a plane earth, a flight along the $x$ axis from 0 to 10 units, no vertical velocity component, and no change in wind velocity with time. Assume $\mathbf{a}_x$ to be directed to the east and $\mathbf{a}_y$ to the north. The wind velocity at the operating altitude is assumed to be:

$$\mathbf{v}(x, y) = \frac{(0.01x^2 - 0.08x + 0.66)\mathbf{a}_x - (0.05x - 0.4)\mathbf{a}_y}{1 + 0.5y^2}$$

Determine the location and magnitude of (*a*) the maximum tailwind encountered; (*b*) repeat for headwind; (*c*) repeat for crosswind; (*d*) Would more favorable tailwinds be available at some other latitude? If so, where?

# 2 CHAPTER

# Coulomb's Law and Electric Field Intensity

Having formulated the language of vector analysis in the first chapter, we next establish and describe a few basic principles of electricity. In this chapter, we introduce Coulomb's electrostatic force law and then formulate this in a general way using field theory. The tools that will be developed can be used to solve any problem in which forces between static charges are to be evaluated or to determine the electric field that is associated with any charge distribution. Initially, we will restrict the study to fields in *vacuum* or *free space*; this would apply to media such as air and other gases. Other materials are introduced in Chapters 5 and 6 and time-varying fields are introduced in Chapter 9. ■

## 2.1 THE EXPERIMENTAL LAW OF COULOMB

Records from at least 600 B.C. show evidence of the knowledge of static electricity. The Greeks were responsible for the term *electricity*, derived from their word for amber, and they spent many leisure hours rubbing a small piece of amber on their sleeves and observing how it would then attract pieces of fluff and stuff. However, their main interest lay in philosophy and logic, not in experimental science, and it was many centuries before the attracting effect was considered to be anything other than magic or a "life force."

Dr. Gilbert, physician to Her Majesty the Queen of England, was the first to do any true experimental work with this effect, and in 1600 he stated that glass, sulfur, amber, and other materials, which he named, would "not only draw to themselves straws and chaff, but all metals, wood, leaves, stone, earths, even water and oil."

Shortly thereafter, an officer in the French Army Engineers, Colonel Charles Coulomb, performed an elaborate series of experiments using a delicate torsion balance, invented by himself, to determine quantitatively the force exerted between two objects, each having a static charge of electricity. His published result bears a great similarity to Newton's gravitational law (discovered about a hundred years earlier).

Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance, which is large compared to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = k\frac{Q_1 Q_2}{R^2}$$

where $Q_1$ and $Q_2$ are the positive or negative quantities of charge, $R$ is the separation, and $k$ is a proportionality constant. If the International System of Units[1] (SI) is used, $Q$ is measured in coulombs (C), $R$ is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality $k$ is written as

$$k = \frac{1}{4\pi\epsilon_0}$$

The new constant $\epsilon_0$ is called the *permittivity of free space* and has magnitude, measured in farads per meter (F/m),

$$\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi}10^{-9} \text{ F/m} \tag{1}$$

The quantity $\epsilon_0$ is not dimensionless, for Coulomb's law shows that it has the label $C^2/N \cdot m^2$. We will later define the farad and show that it has the dimensions $C^2/N \cdot m$; we have anticipated this definition by using the unit F/m in equation (1).

Coulomb's law is now

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \tag{2}$$

The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in SI units as $1.602 \times 10^{-19}$ C; hence a negative charge of one coulomb represents about $6 \times 10^{18}$ electrons.[2] Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is $9 \times 10^9$ N, or about one million tons. The electron has a rest mass of $9.109 \times 10^{-31}$kg and has a radius of the order of magnitude of $3.8 \times 10^{-15}$ m. This does not mean that the electron is spherical in shape, but merely describes the size of the region in which a slowly moving electron has the greatest probability of being found. All other known charged particles, including the proton, have larger masses and larger radii, and occupy a probabilistic volume larger than does the electron.

In order to write the vector form of (2), we need the additional fact (furnished also by Colonel Coulomb) that the force acts along the line joining the two charges

[1] The International System of Units (an mks system) is described in Appendix B. Abbreviations for the units are given in Table B.1. Conversions to other systems of units are given in Table B.2, while the prefixes designating powers of ten in SI appear in Table B.3.

[2] The charge and mass of an electron and other physical constants are tabulated in Table C.4 of Appendix C.

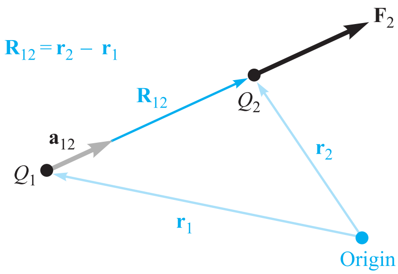


**Figure 2.1** If $Q_1$ and $Q_2$ have like signs, the vector force $\mathbf{F}_2$ on $Q_2$ is in the same direction as the vector $\mathbf{R}_{12}$.

and is repulsive if the charges are alike in sign or attractive if they are of opposite sign. Let the vector $\mathbf{r}_1$ locate $Q_1$, whereas $\mathbf{r}_2$ locates $Q_2$. Then the vector $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ represents the directed line segment from $Q_1$ to $Q_2$, as shown in Figure 2.1. The vector $\mathbf{F}_2$ is the force on $Q_2$ and is shown for the case where $Q_1$ and $Q_2$ have the same sign. The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{3}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of $R_{12}$, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4}$$

**EXAMPLE 2.1**

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5)$ in a vacuum. We desire the force exerted on $Q_2$ by $Q_1$.

***Solution.*** We use (3) and (4) to obtain the vector force. The vector $\mathbf{R}_{12}$ is

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{ N}$$

The magnitude of the force is 30 N, and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on $Q_2$ may also be considered as three component forces,

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction. We might equally well have written

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{5}$$

Coulomb's law is linear, for if we multiply $Q_1$ by a factor $n$, the force on $Q_2$ is also multiplied by the same factor $n$. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

**D2.1.** A charge $Q_A = -20\,\mu\text{C}$ is located at $A(-6, 4, 7)$, and a charge $Q_B = 50\,\mu\text{C}$ is at $B(5, 8, -2)$ in free space. If distances are given in meters, find: (*a*) $\mathbf{R}_{AB}$; (*b*) $R_{AB}$. Determine the vector force exerted on $Q_A$ by $Q_B$ if $\epsilon_0 =$ (*c*) $10^{-9}/(36\pi)$ F/m; (*d*) $8.854 \times 10^{-12}$ F/m.

**Ans.** $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; 14.76 m; $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$ mN; $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$ mN

## 2.2 ELECTRIC FIELD INTENSITY

If we now consider one charge fixed in position, say $Q_1$, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force *field* that is associated with charge, $Q_1$. Call this second charge a test charge $Q_t$. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives the *electric field intensity*, $\mathbf{E}_1$ arising from $Q_1$:

$$\mathbf{E}_1 = \frac{\mathbf{F}_t}{Q_1} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t} \tag{6}$$

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$\mathbf{E}_1$ is interpreted as the vector force, arising from charge $Q_1$, that acts on a unit positive test charge. More generally, we write the defining expression:

$$\mathbf{E} = \frac{\mathbf{F}_t}{Q_t} \tag{7}$$

in which $\mathbf{E}$, a vector function, is the electric field intensity *evaluated at the test charge location* that arises from all *other* charges in the vicinity—meaning the electric field arising from the test charge itself is not included in $\mathbf{E}$.

The units of $\mathbf{E}$ would be in force per unit charge (newtons per coulomb). Again anticipating a new dimensional quantity, the *volt* (V), having the label of joules per coulomb (J/C), or newton-meters per coulomb ($N \cdot m/C$), we measure electric field intensity in the practical units of volts per meter (V/m).

Now, we dispense with most of the subscripts in (6), reserving the right to use them again any time there is a possibility of misunderstanding. The electric field of a single point charge becomes:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R \tag{8}$$

We remember that $R$ is the magnitude of the vector $\mathbf{R}$, the directed line segment from the point at which the point charge $Q$ is located to the point at which $\mathbf{E}$ is desired, and $\mathbf{a}_R$ is a unit vector in the $\mathbf{R}$ direction.[3]

We arbitrarily locate $Q_1$ at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and $R$ is $r$. Hence

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\mathbf{a}_r \tag{9}$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

If we consider a charge that is *not* at the origin of our coordinate system, the field no longer possesses spherical symmetry, and we might as well use rectangular coordinates. For a charge $Q$ located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$, as illustrated in Figure 2.2, we find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ by expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$, and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$
$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \tag{10}$$

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting $\mathbf{E}$ be symbolized in functional notation by $\mathbf{E}(\mathbf{r})$.

Because the coulomb forces are linear, the electric field intensity arising from two point charges, $Q_1$ at $\mathbf{r}_1$ and $Q_2$ at $\mathbf{r}_2$, is the sum of the forces on $Q_t$ caused by $Q_1$ and $Q_2$ acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of $(\mathbf{r}-\mathbf{r}_1)$ and $(\mathbf{r}-\mathbf{r}_2)$, respectively. The vectors $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}-\mathbf{r}_1$, $\mathbf{r}-\mathbf{r}_2$, $\mathbf{a}_1$, and $\mathbf{a}_2$ are shown in Figure 2.3.

[3] We firmly intend to avoid confusing $r$ and $\mathbf{a}_r$ with $R$ and $\mathbf{a}_R$. The first two refer specifically to the spherical coordinate system, whereas $R$ and $\mathbf{a}_R$ do not refer to any coordinate system—the choice is still available to us.

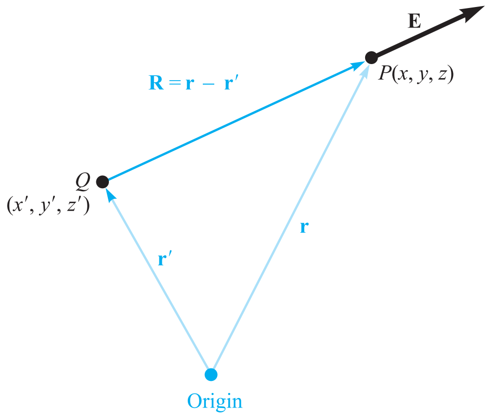


**Figure 2.2** The vector **r**′ locates the point charge $Q$, the vector **r** identifies the general point in space $P(x, y, z)$, and the vector **R** from $Q$ to $P(x, y, z)$ is then $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

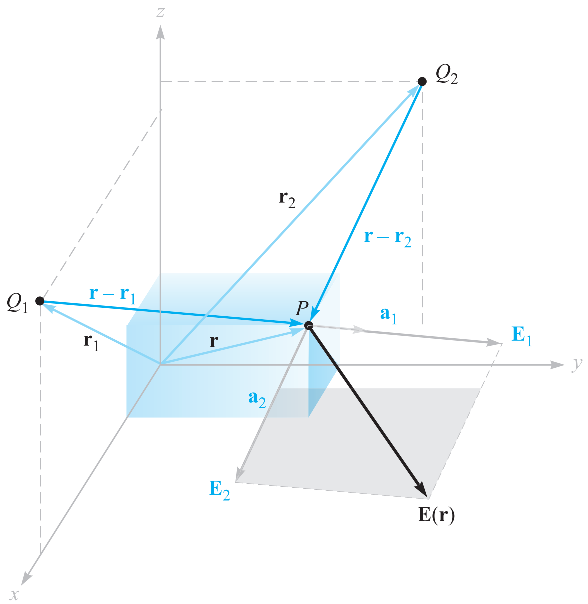


**Figure 2.3** The vector addition of the total electric field intensity at $P$ due to $Q_1$ and $Q_2$ is made possible by the linearity of Coulomb's law.

If we add more charges at other positions, the field due to $n$ point charges is

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$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m \tag{11}$$

## EXAMPLE 2.2

In order to illustrate the application of (11), we find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

***Solution.*** We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r}-\mathbf{r}_1| = 1$, $|\mathbf{r}-\mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r}-\mathbf{r}_3| = 3$, and $|\mathbf{r}-\mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3\times10^{-9}/(4\pi\times8.854\times10^{-12}) = 26.96\ \text{V}\cdot\text{m}$, we may now use (11) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x+\mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x+2\mathbf{a}_y+\mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y+\mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \text{ V/m}$$

**D2.2.** A charge of $-0.3\,\mu$C is located at $A(25, -30, 15)$ (in cm), and a second charge of $0.5\,\mu$C is at $B(-10, 8, 12)$ cm. Find $\mathbf{E}$ at: (*a*) the origin; (*b*) $P(15, 20, 50)$ cm.

**Ans.** $92.3\mathbf{a}_x - 77.6\mathbf{a}_y - 94.2\mathbf{a}_z$ kV/m; $11.9\mathbf{a}_x - 0.519\mathbf{a}_y + 12.4\mathbf{a}_z$ kV/m

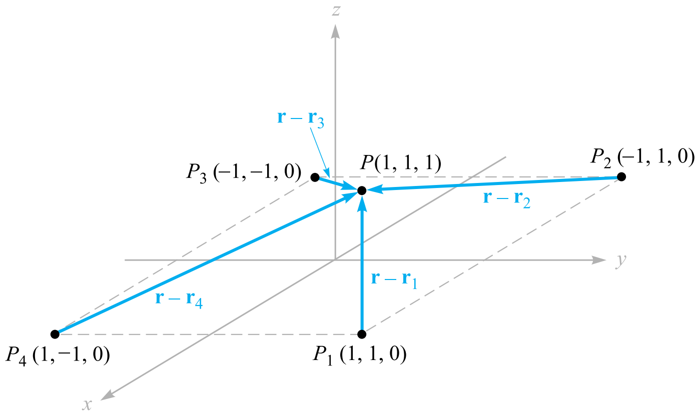



**Figure 2.4** A symmetrical distribution of four identical 3-nC point charges produces a field at $P$, $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

**D2.3.** Evaluate the sums: (*a*) $\sum_{m=0}^{5} \frac{1+(-1)^m}{m^2+1}$; (*b*) $\sum_{m=1}^{4} \frac{(0.1)^m+1}{(4+m^2)^{1.5}}$

**Ans.** 2.52; 0.176

## 2.3 FIELD ARISING FROM A CONTINUOUS VOLUME CHARGE DISTRIBUTION

If we now visualize a region of space filled with a tremendous number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density*, just as we describe water as having a density of 1 $g/cm^3$ (gram per cubic centimeter) even though it consists of atomic- and molecular-sized particles. We can do this only if we are uninterested in the small irregularities (or ripples) in the field as we move from electron to electron or if we care little that the mass of the water actually increases in small but finite steps as each new molecule is added.

This is really no limitation at all, because the end results for electrical engineers are almost always in terms of a current in a receiving antenna, a voltage in an electronic circuit, or a charge on a capacitor, or in general in terms of some large-scale *macroscopic* phenomenon. It is very seldom that we must know a current electron by electron.[4]

We denote volume charge density by $\rho_\nu$, having the units of coulombs per cubic meter ($C/m^3$).

The small amount of charge $\Delta Q$ in a small volume $\Delta\nu$ is

$$\Delta Q = \rho_\nu \Delta\nu \tag{12}$$

and we may define $\rho_\nu$ mathematically by using a limiting process on (12),

$$\rho_\nu = \lim_{\Delta\nu \to 0} \frac{\Delta Q}{\Delta\nu} \tag{13}$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{\text{vol}} \rho_\nu d\nu \tag{14}$$

Only one integral sign is customarily indicated, but the differential $d\nu$ signifies integration throughout a volume, and hence a triple integration.

[4] A study of the noise generated by electrons in semiconductors and resistors, however, requires just such an examination of the charge through statistical analysis.

**EXAMPLE 2.3**

As an example of the evaluation of a volume integral, we find the total charge contained in a 2-cm length of the electron beam shown in Figure 2.5.

***Solution.*** From the illustration, we see that the charge density is

$$\rho_v = -5 \times 10^{-6} e^{-10^5 \rho z} \text{ C/m}^2$$

The volume differential in cylindrical coordinates is given in Section 1.8; therefore,

$$Q = \int_{0.02}^{0.04} \int_0^{2\pi} \int_0^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z} \rho \, d\rho \, d\phi \, dz$$

We integrate first with respect to $\phi$ because it is so easy,

$$Q = \int_{0.02}^{0.04} \int_0^{0.01} -10^{-5} \pi e^{-10^5 \rho z} \rho \, d\rho \, dz$$

and then with respect to $z$, because this will simplify the last integration with respect to $\rho$,

$$Q = \int_0^{0.01} \left( \frac{-10^{-5}\pi}{-10^5 \rho} e^{-10^5 \rho z} \rho \, d\rho \right)_{z=0.02}^{z=0.04}$$

$$= \int_0^{0.01} -10^{-5} \pi (e^{-2000\rho} - e^{-4000\rho}) d\rho$$

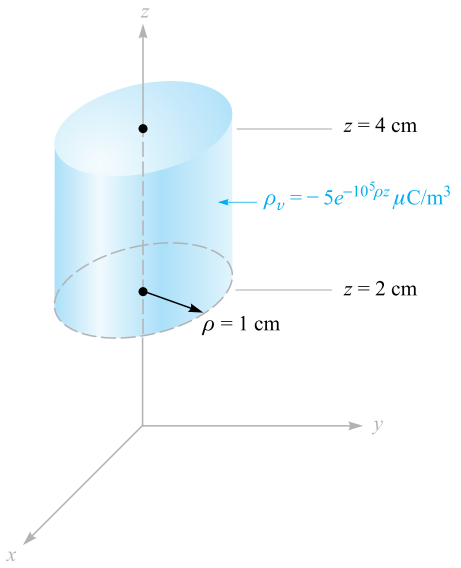



**Figure 2.5** The total charge contained within the right circular cylinder may be obtained by evaluating $Q = \int_{\text{vol}} \rho_v dv$.

Finally,

$$Q = -10^{-10}\pi \left( \frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000} \right)_0^{0.01}$$

$$Q = -10^{-10}\pi \left( \frac{1}{2000} - \frac{1}{4000} \right) = \frac{-\pi}{40} = 0.0785 \text{ pC}$$

where pC indicates picocoulombs.

The incremental contribution to the electric field intensity at $\mathbf{r}$ produced by an incremental charge $\Delta Q$ at $\mathbf{r}'$ is

$$\Delta\mathbf{E}(\mathbf{r}) = \frac{\Delta Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\rho_\nu \Delta\nu}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

If we sum the contributions of all the volume charge in a given region and let the volume element $\Delta\nu$ approach zero as the number of these elements becomes infinite, the summation becomes an integral,

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_\nu(\mathbf{r}')\,d\nu'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{15}$$

This is again a triple integral, and (except in Drill Problem 2.4) we shall do our best to avoid actually performing the integration.

The significance of the various quantities under the integral sign of (15) might stand a little review. The vector $\mathbf{r}$ from the origin locates the field point where $\mathbf{E}$ is being determined, whereas the vector $\mathbf{r}'$ extends from the origin to the source point where $\rho_\nu(\mathbf{r}')d\nu'$ is located. The scalar distance between the source point and the field point is $|\mathbf{r}-\mathbf{r}'|$, and the fraction $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ is a unit vector directed from source point to field point. The variables of integration are $x'$, $y'$, and $z'$ in rectangular coordinates.

**D2.4.** Calculate the total charge within each of the indicated volumes: (*a*) $0.1 \le |x|, |y|, |z| \le 0.2$: $\rho_\nu = \dfrac{1}{x^3y^3z^3}$; (*b*) $0 \le \rho \le 0.1, 0 \le \phi \le \pi, 2 \le z \le 4$; $\rho_\nu = \rho^2 z^2 \sin 0.6\phi$; (*c*) universe: $\rho_\nu = e^{-2r}/r^2$.

**Ans.** 0; 1.018 mC; 6.28 C

## 2.4 FIELD OF A LINE CHARGE

Up to this point we have considered two types of charge distribution, the point charge and charge distributed throughout a volume with a density $\rho_\nu$ C/m$^3$. If we now consider a filamentlike distribution of volume charge density, such as a charged conductor of very small radius, we find it convenient to treat the charge as a line charge of density $\rho_L$ C/m.

We assume a straight-line charge extending along the $z$ axis in a cylindrical coordinate system from $-\infty$ to $\infty$, as shown in Figure 2.6. We desire the electric field intensity $\mathbf{E}$ at any and every point resulting from a *uniform* line charge density $\rho_L$.

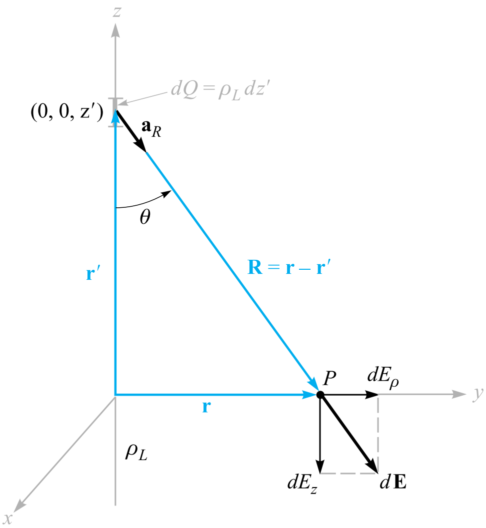



**Figure 2.6** The contribution $d\mathbf{E} = dE_\rho \mathbf{a}_\rho + dE_z \mathbf{a}_z$ to the electric field intensity produced by an element of charge $dQ = \rho_L dz'$ located a distance $z'$ from the origin. The linear charge density is uniform and extends along the entire $z$ axis.

Symmetry should always be considered first in order to determine two specific factors: (1) with which coordinates the field does *not* vary, and (2) which components of the field are *not* present. The answers to these questions then tell us which components are present and with which coordinates they *do* vary.

Referring to Figure 2.6, we realize that as we move around the line charge, varying $\phi$ while keeping $\rho$ and $z$ constant, the line charge appears the same from every angle. In other words, azimuthal symmetry is present, and no field component may vary with $\phi$.

Again, if we maintain $\rho$ and $\phi$ constant while moving up and down the line charge by changing $z$, the line charge still recedes into infinite distance in both directions and the problem is unchanged. This is axial symmetry and leads to fields that are not functions of $z$.

If we maintain $\phi$ and $z$ constant and vary $\rho$, the problem changes, and Coulomb's law leads us to expect the field to become weaker as $\rho$ increases. Hence, by a process of elimination we are led to the fact that the field varies only with $\rho$.

Now, which components are present? Each incremental length of line charge acts as a point charge and produces an incremental contribution to the electric field intensity which is directed away from the bit of charge (assuming a positive line charge). No element of charge produces a $\phi$ component of electric intensity; $E_\phi$ is zero. However, each element does produce an $E_\rho$ and an $E_z$ component, but the contribution to $E_z$ by elements of charge that are equal distances above and below the point at which we are determining the field will cancel.

We therefore have found that we have only an $E_\rho$ component and it varies only with $\rho$. Now to find this component.

We choose a point $P(0, y, 0)$ on the $y$ axis at which to determine the field. This is a perfectly general point in view of the lack of variation of the field with $\phi$ and $z$. Applying (10) to find the incremental field at $P$ due to the incremental charge $dQ = \rho_L dz'$, we have

$$d\mathbf{E} = \frac{\rho_L dz'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$

where

$$\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$$
$$\mathbf{r}' = z'\mathbf{a}_z$$

and

$$\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

Therefore,

$$d\mathbf{E} = \frac{\rho_L dz'(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Because only the $\mathbf{E}_\rho$ component is present, we may simplify:

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

and

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Integrating by integral tables or change of variable, $z' = \rho \cot\theta$, we have

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0}\rho\left(\frac{1}{\rho^2}\frac{z'}{\sqrt{\rho^2 + z'^2}}\right)_{-\infty}^{\infty}$$

and

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

or finally,

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho} \qquad (16)$$

We note that the field falls off inversely with the distance to the charged line, as compared with the point charge, where the field decreased with the *square* of the distance. Moving ten times as far from a point charge leads to a field only 1 percent the previous strength, but moving ten times as far from a line charge only reduces the field to 10 percent of its former value. An analogy can be drawn with a source of

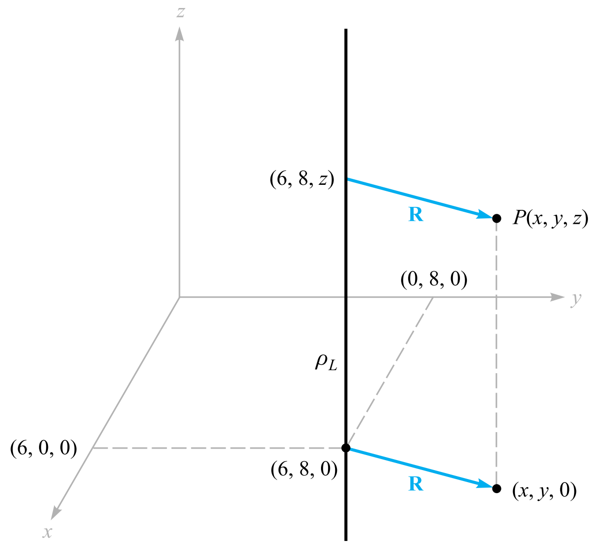



**Figure 2.7** A point $P(x, y, z)$ is identified near an infinite uniform line charge located at $x = 6$, $y = 8$.

illumination, for the light intensity from a point source of light also falls off inversely as the square of the distance to the source. The field of an infinitely long fluorescent tube thus decays inversely as the first power of the radial distance to the tube, and we should expect the light intensity about a finite-length tube to obey this law near the tube. As our point recedes farther and farther from a finite-length tube, however, it eventually looks like a point source, and the field obeys the inverse-square relationship.

Before leaving this introductory look at the field of the infinite line charge, we should recognize the fact that not all line charges are located along the $z$ axis. As an example, let us consider an infinite line charge parallel to the $z$ axis at $x = 6$, $y = 8$, shown in Figure 2.7. We wish to find $\mathbf{E}$ at the general field point $P(x, y, z)$.

We replace $\rho$ in (16) by the radial distance between the line charge and point, $P$, $R = \sqrt{(x-6)^2 + (y-8)^2}$, and let $\mathbf{a}_\rho$ be $\mathbf{a}_R$. Thus,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\sqrt{(x-6)^2 + (y-8)^2}}\mathbf{a}_R$$

where

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

Therefore,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0}\frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{(x-6)^2 + (y-8)^2}$$

We again note that the field is not a function of $z$.

In Section 2.6, we describe how fields may be sketched, and we use the field of the line charge as one example.

**D2.5.** Infinite uniform line charges of 5 nC/m lie along the (positive and negative) $x$ and $y$ axes in free space. Find $\mathbf{E}$ at: ($a$) $P_A(0, 0, 4)$; ($b$) $P_B(0, 3, 4)$.

**Ans.** $45\mathbf{a}_z$ V/m; $10.8\mathbf{a}_y + 36.9\mathbf{a}_z$ V/m

## 2.5 FIELD OF A SHEET OF CHARGE

Another basic charge configuration is the infinite sheet of charge having a uniform density of $\rho_S$ C/m$^2$. Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor. As we shall see in Chapter 5, static charge resides on conductor surfaces and not in their interiors; for this reason, $\rho_S$ is commonly known as *surface charge density*. The charge-distribution family now is complete—point, line, surface, and volume, or $Q$, $\rho_L$, $\rho_S$, and $\rho_v$.

Let us place a sheet of charge in the $yz$ plane and again consider symmetry (Figure 2.8). We see first that the field cannot vary with $y$ or with $z$, and then we see that the $y$ and $z$ components arising from differential elements of charge symmetrically located with respect to the point at which we evaluate the field will cancel. Hence only $E_x$ is present, and this component is a function of $x$ alone. We are again faced with a choice of many methods by which to evaluate this component, and this time we use only one method and leave the others as exercises for a quiet Sunday afternoon.

Let us use the field of the infinite line charge (16) by dividing the infinite sheet into differential-width strips. One such strip is shown in Figure 2.8. The line charge

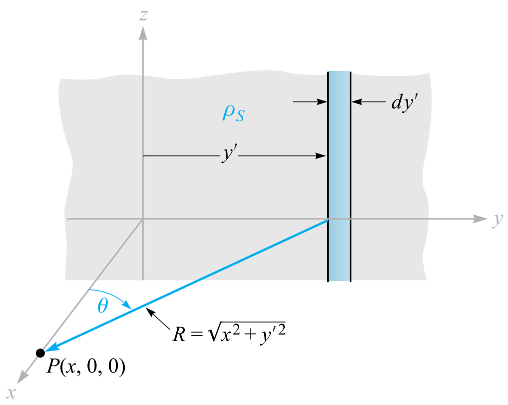



**Figure 2.8** An infinite sheet of charge in the $yz$ plane, a general point $P$ on the $x$ axis, and the differential-width line charge used as the element in determining the field at $P$ by $d\mathbf{E} = \rho_S\, dy' \mathbf{a}_R/(2\pi\varepsilon_0 R)$.

density, or charge per unit length, is $\rho_L = \rho_S \, dy'$, and the distance from this line charge to our general point $P$ on the $x$ axis is $R = \sqrt{x^2 + y'^2}$. The contribution to $E_x$ at $P$ from this differential-width strip is then

$$dE_x = \frac{\rho_S \, dy'}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \cos\theta = \frac{\rho_S}{2\pi\epsilon_0} \frac{x dy'}{x^2 + y'^2}$$

Adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x \, dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0} \tan^{-1} \frac{y'}{x} \Big]_{-\infty}^{\infty} = \frac{\rho_S}{2\epsilon_0}$$

If the point $P$ were chosen on the negative $x$ axis, then

$$E_x = -\frac{\rho_S}{2\epsilon_0}$$

for the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector $\mathbf{a}_N$, which is normal to the sheet and directed outward, or away from it. Then

$$\mathbf{E} = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_N \tag{17}$$

This is a startling answer, for the field is constant in magnitude and direction. It is just as strong a million miles away from the sheet as it is right off the surface. Returning to our light analogy, we see that a uniform source of light on the ceiling of a very large room leads to just as much illumination on a square foot on the floor as it does on a square foot a few inches below the ceiling. If you desire greater illumination on this subject, it will do you no good to hold the book closer to such a light source.

If a second infinite sheet of charge, having a *negative* charge density $-\rho_S$, is located in the plane $x = a$, we may find the total field by adding the contribution of each sheet. In the region $x > a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E}_- = -\frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and for $x < 0$,

$$\mathbf{E}_+ = -\frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = 0$$

and when $0 < x < a$,

$$\mathbf{E}_+ = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x \qquad \mathbf{E}_- = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_x$$

and

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho_S}{\epsilon_0} \mathbf{a}_x \tag{18}$$

This is an important practical answer, for it is the field between the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation and provided also that we are considering a point well removed from the edges. The field outside the capacitor, while not zero, as we found for the preceding ideal case, is usually negligible.

**D2.6.** Three infinite uniform sheets of charge are located in free space as follows: 3 nC/m$^2$ at $z = -4$, 6 nC/m$^2$ at $z = 1$, and $-8$ nC/m$^2$ at $z = 4$. Find **E** at the point: (*a*) $P_A(2, 5, -5)$; (*b*) $P_B(4, 2, -3)$; (*c*) $P_C(-1, -5, 2)$; (*d*) $P_D(-2, 4, 5)$.

**Ans.** $-56.5\mathbf{a}_z$; $283\mathbf{a}_z$; $961\mathbf{a}_z$; $56.5\mathbf{a}_z$ all V/m

## 2.6 STREAMLINES AND SKETCHES OF FIELDS

We now have vector equations for the electric field intensity resulting from several different charge configurations, and we have had little difficulty in interpreting the magnitude and direction of the field from the equations. Unfortunately, this simplicity cannot last much longer, for we have solved most of the simple cases and our new charge distributions must lead to more complicated expressions for the fields and more difficulty in visualizing the fields through the equations. However, it is true that one picture would be worth about a thousand words, if we just knew what picture to draw.

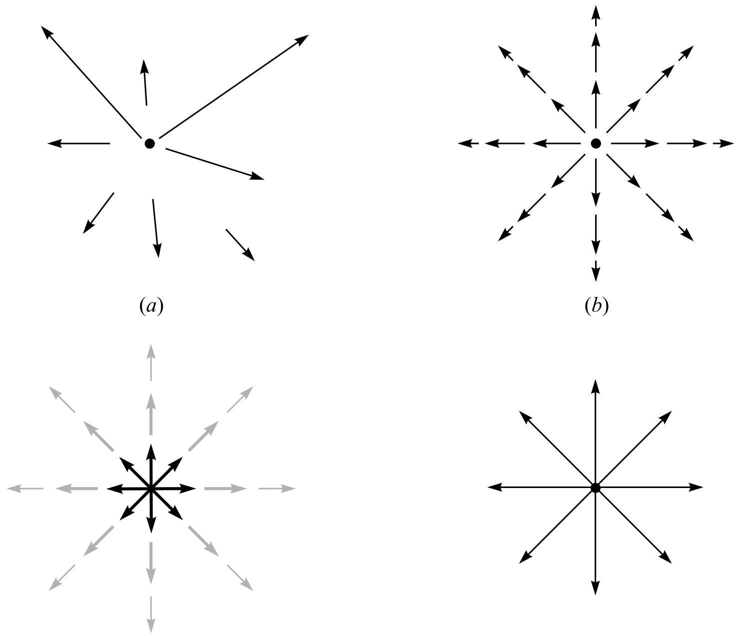
Consider the field about the line charge,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

Figure 2.9*a* shows a cross-sectional view of the line charge and presents what might be our first effort at picturing the field—short line segments drawn here and there having lengths proportional to the magnitude of **E** and pointing in the direction of **E**. The figure fails to show the symmetry with respect to $\phi$, so we try again in Figure 2.9*b* with a symmetrical location of the line segments. The real trouble now appears—the longest lines must be drawn in the most crowded region, and this also plagues us if we use line segments of equal length but of a thickness that is proportional to **E** (Figure 2.9*c*). Other schemes include drawing shorter lines to represent stronger fields (inherently misleading) and using intensity of color or different colors to represent stronger fields.

For the present, let us be content to show only the *direction* of **E** by drawing continuous lines, which are everywhere tangent to **E**, from the charge. Figure 2.9*d* shows this compromise. A symmetrical distribution of lines (one every 45°) indicates azimuthal symmetry, and arrowheads should be used to show direction.

These lines are usually called *streamlines*, although other terms such as flux lines and direction lines are also used. A small positive test charge placed at any point in this field and free to move would accelerate in the direction of the streamline passing through that point. If the field represented the velocity of a liquid or a gas (which, incidentally, would have to have a source at $\rho = 0$), small suspended particles in the liquid or gas would trace out the streamlines.



(*a*) (*b*)

(*c*) (*d*)

**Figure 2.9** (*a*) One very poor sketch, (*b*) and (*c*) two fair sketches, and (*d* ) the usual form of a streamline sketch. In the last form, the arrows show the direction of the field at every point along the line, and the spacing of the lines is inversely proportional to the strength of the field.

We will find out later that a bonus accompanies this streamline sketch, for the magnitude of the field can be shown to be inversely proportional to the spacing of the streamlines for some important special cases. The closer they are together, the stronger is the field. At that time we will also find an easier, more accurate method of making that type of streamline sketch.

If we attempted to sketch the field of the point charge, the variation of the field into and away from the page would cause essentially insurmountable difficulties; for this reason sketching is usually limited to two-dimensional fields.

In the case of the two-dimensional field, let us arbitrarily set $E_z = 0$. The streamlines are thus confined to planes for which $z$ is constant, and the sketch is the same for any such plane. Several streamlines are shown in Figure 2.10, and the $E_x$ and $E_y$ components are indicated at a general point. It is apparent from the geometry that

$$\frac{E_y}{E_x} = \frac{dy}{dx} \tag{19}$$

A knowledge of the functional form of $E_x$ and $E_y$ (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

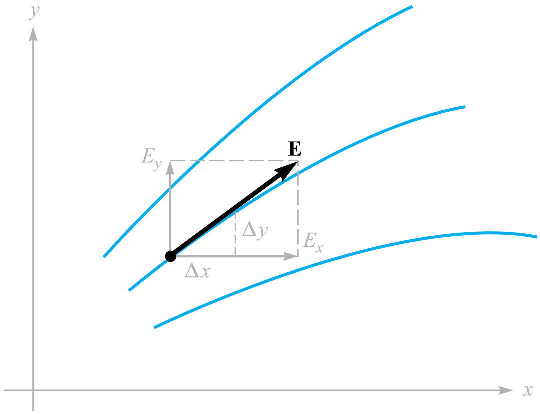



**Figure 2.10** The equation of a streamline is obtained by solving the differential equation $E_y/E_x = dy/dx$.

As an illustration of this method, consider the field of the uniform line charge with $\rho_L = 2\pi\epsilon_0$,

$$\mathbf{E} = \frac{1}{\rho}\mathbf{a}_\rho$$

In rectangular coordinates,

$$\mathbf{E} = \frac{x}{x^2+y^2}\mathbf{a}_x + \frac{y}{x^2+y^2}\mathbf{a}_y$$

Thus we form the differential equation

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y}{x} \qquad \text{or} \qquad \frac{dy}{y} = \frac{dx}{x}$$

Therefore,

$$\ln y = \ln x + C_1 \qquad \text{or} \qquad \ln y = \ln x + \ln C$$

from which the equations of the streamlines are obtained,

$$y = Cx$$

If we want to find the equation of one particular streamline, say one passing through $P(-2, 7, 10)$, we merely substitute the coordinates of that point into our equation and evaluate $C$. Here, $7 = C(-2)$, and $C = -3.5$, so $y = -3.5x$.

Each streamline is associated with a specific value of $C$, and the radial lines shown in Figure 2.9*d* are obtained when $C = 0, 1, -1$, and $1/C = 0$.

The equations of streamlines may also be obtained directly in cylindrical or spherical coordinates. A spherical coordinate example will be examined in Section 4.7.

**D2.7.** Find the equation of that streamline that passes through the point $P(1, 4, -2)$ in the field $\mathbf{E} =$ (*a*) $\dfrac{-8x}{y}\mathbf{a}_x + \dfrac{4x^2}{y^2}\mathbf{a}_y$; (*b*) $2e^{5x}[y(5x+1)\mathbf{a}_x + x\mathbf{a}_y]$.

**Ans.** $x^2 + 2y^2 = 33$; $y^2 = 15.7 + 0.4x - 0.08\ln(5x+1)$

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3. Schelkunoff, S. A. *Electromagnetic Fields*. New York: Blaisdell Publishing Company, 1963. Many of the physical aspects of fields are discussed early in this text without advanced mathematics.


## CHAPTER 2 PROBLEMS

**2.1** Three point charges are positioned in the $x$-$y$ plane as follows: 5 nC at $y = 5$ cm, $-10$ nC at $y = -5$ cm, and 15 nC at $x = -5$ cm. Find the required $x$-$y$ coordinates of a 20-nC fourth charge that will produce a zero electric field at the origin.

**2.2** Point charges of 1 nC and $-2$ nC are located at (0, 0, 0) and (1, 1, 1), respectively, in free space. Determine the vector force acting on each charge.

**2.3** Point charges of 50 nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at $A$.

**2.4** Eight identical point charges of $Q$ C each are located at the corners of a cube of side length $a$, with one charge at the origin, and with the three nearest charges at $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$. Find an expression for the total vector force on the charge at $P(a, a, a)$, assuming free space.

**2.5** Let a point charge $Q_1 = 25$ nC be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3, 4, -2)$. (*a*) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1, 2, 3)$. (*b*) At what point on the $y$ axis is $E_x = 0$?

**2.6** Two point charges of equal magnitude $q$ are positioned at $z = \pm d/2$. (*a*) Find the electric field everywhere on the $z$ axis; (*b*) find the electric field everywhere on the $x$ axis; (*c*) repeat parts (*a*) and (*b*) if the charge at $z = -d/2$ is $-q$ instead of $+q$.

**2.7** A 2-$\mu$C point charge is located at $A(4, 3, 5)$ in free space. Find $E_\rho$, $E_\phi$, and $E_z$ at $P(8, 12, 2)$.

**2.8** A crude device for measuring charge consists of two small insulating spheres of radius $a$, one of which is fixed in position. The other is movable along the $x$ axis and is subject to a restraining force $kx$, where $k$ is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of $Q$/C, obtain the expression by which $Q$ may be found as a function of $x$. Determine the maximum charge that can be measured in terms of $\epsilon_0$, $k$, and $d$, and state the separation of the spheres then. What happens if a larger charge is applied?

**2.9** A 100-nC point charge is located at $A(-1, 1, 3)$ in free space. (*a*) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m. (*b*) Find $y_1$ if $P(-2, y_1, 3)$ lies on that locus.

**2.10** A charge of $-1$ nC is located at the origin in free space. What charge must be located at (2, 0, 0) to cause $E_x$ to be zero at (3, 1, 1)?

**2.11** A charge $Q_0$ located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$. (*a*) Find $Q_0$. Find **E** at $M(1, 6, 5)$ in (*b*) rectangular coordinates; (*c*) cylindrical coordinates; (*d*) spherical coordinates.

**2.12** Electrons are in random motion in a fixed region in space. During any 1 $\mu$s interval, the probability of finding an electron in a subregion of volume $10^{-15}$ m$^2$ is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

**2.13** A uniform volume charge density of 0.2 $\mu$C/m$^3$ is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere, find (*a*) the total charge present throughout the shell, and (*b*) $r_1$ if half the total charge is located in the region 3 cm $< r < r_1$.

**2.14** The electron beam in a certain cathode ray tube possesses cylindrical symmetry, and the charge density is represented by $\rho_v = -0.1/(\rho^2 + 10^{-8})$ pC/m$^3$ for $0 < \rho < 3 \times 10^{-4}$ m, and $\rho_v = 0$ for $\rho > 3 \times 10^{-4}$ m. (*a*) Find the total charge per meter along the length of the beam; (*b*) if the electron velocity is $5 \times 10^7$ m/s, and with one ampere defined as 1C/s, find the beam current.

**2.15** A spherical volume having a 2-$\mu$m radius contains a uniform volume charge density of $10^{15}$ C/m$^3$. (*a*) What total charge is enclosed in the spherical volume? (*b*) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3 mm on a side and that there is no charge between the spheres. What is the average volume charge density throughout this large region?

**2.16** Within a region of free space, charge density is given as $\rho_v = \frac{\rho_0 r \cos\theta}{a}$ C/m$^3$, where $\rho_0$ and $a$ are constants. Find the total charge lying within (*a*) the sphere, $r \leq a$; (*b*) the cone, $r \leq a$, $0 \leq \theta \leq 0.1\pi$; (*c*) the region, $r \leq a$, $0 \leq \theta \leq 0.1\pi$, $0 \leq \phi \leq 0.2\pi$.

**2.17** A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$: (*a*) find **E** at $P(1, 2, 3)$. (*b*) find **E** at that point in the $z = 0$ plane where the direction of **E** is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$.

**2.18** (a) Find **E** in the plane $z = 0$ that is produced by a uniform line charge, $\rho_L$, extending along the $z$ axis over the range $-L < z < L$ in a cylindrical coordinate system. (b) If the finite line charge is approximated by an infinite line charge ($L \to \infty$), by what percentage is $E_\rho$ in error if $\rho = 0.5L$? (*c*) Repeat (*b*) with $\rho = 0.1L$.

**2.19** A uniform line charge of 2 $\mu$C/m is located on the $z$ axis. Find **E** in rectangular coordinates at $P(1, 2, 3)$ if the charge exists from (*a*) $-\infty < z < \infty$; (*b*) $-4 \le z \le 4$.

**2.20** A line charge of uniform charge density $\rho_0$ C/m and of length $\ell$ is oriented along the $z$ axis at $-\ell/2 < z < \ell/2$. (*a*) Find the electric field strength, **E**, in magnitude and direction at any position along the $x$ axis. (*b*) With the given line charge in position, find the force acting on an identical line charge that is oriented along the $x$ axis at $\ell/2 < x < 3\ell/2$.

**2.21** Two identical uniform line charges, with $\rho_l = 75$ nC/m, are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other?

**2.22** Two identical uniform sheet charges with $\rho_s = 100$ nC/m$^2$ are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

**2.23** Given the surface charge density, $\rho_s = 2\ \mu$C/m$^2$, existing in the region $\rho < 0.2$ m, $z = 0$, find **E** at (*a*) $P_A(\rho = 0, z = 0.5)$; (*b*) $P_B(\rho = 0, z = -0.5)$. Show that (*c*) the field along the $z$ axis reduces to that of an infinite sheet charge at small values of $z$; (*d*) the $z$ axis field reduces to that of a point charge at large values of $z$.

**2.24** (*a*) Find the electric field on the $z$ axis produced by an annular ring of uniform surface charge density $\rho_s$ in free space. The ring occupies the region $z = 0$, $a \le \rho \le b$, $0 \le \phi \le 2\pi$ in cylindrical coordinates. (*b*) From your part (*a*) result, obtain the field of an infinite uniform sheet charge by taking appropriate limits.

**2.25** Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC, at $P(2, 0, 6)$; uniform line charge density, 3 nC/m, at $x = -2$, $y = 3$; uniform surface charge density, 0.2 nC/m$^2$ at $x = 2$.

**2.26** A radially dependent surface charge is distributed on an infinite flat sheet in the $x$-$y$ plane and is characterized in cylindrical coordinates by surface density $\rho_s = \rho_0/\rho$, where $\rho_0$ is a constant. Determine the electric field strength, **E**, everywhere on the $z$ axis.

**2.27** Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find (*a*) the equation of the streamline that passes through the point $P(2, 3, -4)$; (*b*) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$.

**2.28** An electric dipole (discussed in detail in Section 4.7) consists of two point charges of equal and opposite magnitude $\pm Q$ spaced by distance $d$. With the charges along the $z$ axis at positions $z = \pm d/2$ (with the positive charge at the positive $z$ location), the electric field in spherical coordinates is given by $\mathbf{E}(r, \theta) = [Qd/(4\pi\epsilon_0 r^3)][2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude $q$ (*a*) at $(0, 0, z)$; (*b*) at $(0, y, 0)$.

**2.29** If $\mathbf{E} = 20e^{-5y}(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, find (*a*) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$; (*b*) a unit vector in the direction of $\mathbf{E}$ at $P$; (*c*) the equation of the direction line passing through $P$.

**2.30** For fields that do not vary with $z$ in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_p/E_\phi = d\rho/(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30^\circ, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\mathbf{a}_\rho - \rho\sin 2\phi\mathbf{a}_\phi$.

3

CHAPTER

# Electric Flux Density, Gauss's Law, and Divergence

After drawing a few of the fields described in the previous chapter and becoming familiar with the concept of the streamlines that show the direction of the force on a test charge at every point, it is difficult to avoid giving these lines a physical significance and thinking of them as *flux* lines. No physical particle is projected radially outward from the point charge, and there are no steel tentacles reaching out to attract or repel an unwary test charge, but as soon as the streamlines are drawn on paper there seems to be a picture showing "something" is present.

It is very helpful to invent an *electric flux* that streams away symmetrically from a point charge and is coincident with the streamlines and to visualize this flux wherever an electric field is present.

This chapter introduces and uses the concept of electric flux and electric flux density to again solve several of the problems presented in Chapter 2. The work here turns out to be much easier, and this is due to the extremely symmetrical problems that we are solving. ■

## 3.1 ELECTRIC FLUX DENSITY

About 1837, the director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the past ten years when he was experimenting in his now-famous work on induced electromotive force, which we will discuss in Chapter 10. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply *dielectric*) that would occupy the entire volume between the concentric spheres. We will immediately use his findings about dielectric materials, for we are restricting our attention to fields in free space until Chapter 6. At that time we will see that the materials he used will be classified as ideal dielectrics.

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement flux,* or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by $\Psi$ (psi) and the total charge on the inner sphere by $Q$, then for Faraday's experiment

$$\Psi = Q$$

and the electric flux $\Psi$ is measured in coulombs.

We can obtain more quantitative information by considering an inner sphere of radius $a$ and an outer sphere of radius $b$, with charges of $Q$ and $-Q$, respectively (Figure 3.1). The paths of electric flux $\Psi$ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, $\Psi$ coulombs of electric flux are produced by the charge $Q(=\Psi)$ Cs distributed uniformly over a surface having an area of $4\pi a^2$ m$^2$. The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2$ C/m$^2$, and this is an important new quantity.

*Electric flux density*, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we will use the term consistently.

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric