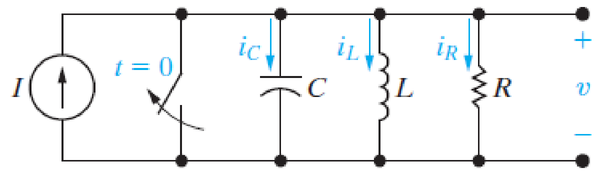
Lecture - 7 -

➢ Step response of a Parallel RLC



$$i_L + i_R + i_C = I,$$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I.$$

$$v = L\frac{di_L}{dt},$$

$$\frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}.$$

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

we divide through by *LC* and rearrange terms:

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

$$\boxed{\begin{aligned} i_L &= I + A_1' e^{s_1 t} + A_2' e^{s_2 t}, \\ i_L &= I + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t, \\ i_L &= I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, \end{aligned}}$$

Example

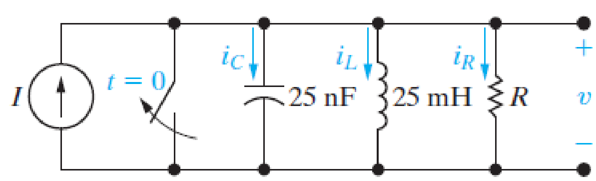
The initial energy stored in the circuit in Fig. is zero. At t=0, a dc current source of 24 mA is applied to the circuit. The value of the resistor is 400Ω.

a) What is the initial value of i_L?

b) What is the initial value of di_L/dt?

c) What are the roots of the characteristic equation?

d) What is the numerical expression for $i_L(t)$ when $t \geq 0$?



Solution

a) No energy is stored in the circuit prior to the application of the dc current source, so the initial current in the inductor is zero. The inductor prohibits an instantaneous change in inductor current; therefore iL(0) = 0, immediately after the switch has been opened.

b) The initial voltage on the capacitor is zero before the switch has been opened; therefore, it will be zero immediately after. Now, because

$$v = L\frac{di_L}{dt},$$

c) From the circuit elements, we obtain

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8,$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(400)(25)} = 5 \times 10^4 \text{ rad/s},$$

or

$$\alpha^2 = 25 \times 10^8.$$

Because $\omega_0^2 < \alpha^2$, the roots of the characteristic equation are real and distinct. Thus

$$s_1 = -5 \times 10^4 + 3 \times 10^4 = -20{,}000 \text{ rad/s},$$

$$s_2 = -5 \times 10^4 - 3 \times 10^4 = -80{,}000 \text{ rad/s}.$$

d)

$$i_L = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}.$$

$$i_L(0) = I_f + A_1' + A_2' = 0,$$

$$\frac{di_L}{dt}(0) = s_1 A_1' + s_2 A_2' = 0.$$

Solving for A_1' and A_2' gives

$$A_1' = -32 \text{ mA} \quad \text{and} \quad A_2' = 8 \text{ mA}.$$

The numerical solution for $i_L(t)$ is

$$i_L(t) = (24 - 32e^{-20{,}000t} + 8e^{-80{,}000t}) \text{ mA}, \quad t \geq 0.$$

- **The Resistor in the Previous Example is increased to 625 Ω . Find $i_L(t)$ for $t \geq 0$.**

Solution

Because L and C remain fixed, ω_0^2 has the same value as in Example that is, $\omega_0^2 = 16 \times 10^8$. Increasing R to 625 Ω decreases α to 3.2×10^4 rad/s. With $\omega_0^2 > \alpha^2$, the roots of the characteristic equation are complex. Hence

$$s_1 = -3.2 \times 10^4 + j2.4 \times 10^4 \text{ rad/s},$$

$$s_2 = -3.2 \times 10^4 - j2.4 \times 10^4 \text{ rad/s}.$$

The current response is now underdamped and given by Eq.

$$i_L(t) = I_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t.$$

Here, α is 32,000 rad/s, ω_d is 24,000 rad/s, and I_f is 24 mA.

As in Example , B_1' and B_2' are determined from the initial conditions. Thus the two simultaneous equations are

$$i_L(0) = I_f + B_1' = 0,$$

$$\frac{di_L}{dt}(0) = \omega_d B_2' - \alpha B_1' = 0.$$

Then,

$$B_1' = -24 \text{ mA}$$

$$B_2' = -32 \text{ mA}.$$

The numerical solution for $i_L(t)$ is

$$i_L(t) = (24 - 24e^{-32{,}000t} \cos 24{,}000t - 32e^{-32{,}000t} \sin 24{,}000t) \text{ mA}, \quad t \geq 0.$$

♦ **The Resistor in the Previous Example is 500 Ω . Find $i_L(t)$ for $t \geq 0$.**

Solution

We know that ω_0^2 remains at 16×10^8. With R set at 500 Ω, α becomes $4 \times 10^4\ \mathrm{s}^{-1}$, which corresponds to critical damping. Therefore the solution for $i_L(t)$ takes the form of Eq.

$$i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}.$$

- ✓ **Inductor current in critically damped parallel RLC circuit step response**

Again, D_1' and D_2' are computed from initial conditions, or

$$i_L(0) = I_f + D_2' = 0,$$

$$\frac{di_L}{dt}(0) = D_1' - \alpha D_2' = 0.$$

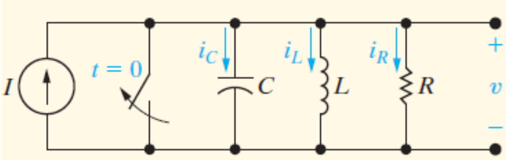
Thus

$$D_1' = -960{,}000\ \mathrm{mA/s} \quad \text{and} \quad D_2' = -24\ \mathrm{mA}.$$

The numerical expression for $i_L(t)$ is

$$i_L(t) = (24 - 960{,}000 t e^{-40{,}000t} - 24e^{-40{,}000t})\ \mathrm{mA},\ t \geq 0.$$

In the circuit shown, $R = 500\ \Omega$, $L = 0.64$ H, $C = 1\ \mu$F, and $I = -1$ A. The initial voltage drop across the capacitor is 40 V and the initial inductor current is 0.5 A. Find (a) $i_R(0^+)$; (b) $i_C(0^+)$; (c) $di_L(0^+)/dt$; (d) s_1, s_2; (e) $i_L(t)$ for $t \geq 0$; and (f) $v(t)$ for $t \geq 0^+$.



Answer: (a) 80 mA;

(b) −1.58 A;

(c) 62.5 A/s;

(d) $(-1000 + j750)$ rad/s, $(-1000 - j750)$ rad/s;

(e) $[-1 + e^{-1000t}[1.5 \cos 750t + 2.0833 \sin 750t]$ A, for $t \geq 0$;

(f) $e^{-1000t}(40 \cos 750t - 2053.33 \sin 750t)$ V, for $t \geq 0^+$.

8.27 Assume that at the instant the 2A dc current source is applied to the circuit in Fig. P8.27, the initial current in the 25 mH inductor is 1 A, and the initial voltage on the capacitor is 50 V (positive at the upper terminal). Find the expression for $i_L(t)$ for $t \geq 0$ if R equals 12.5 Ω.

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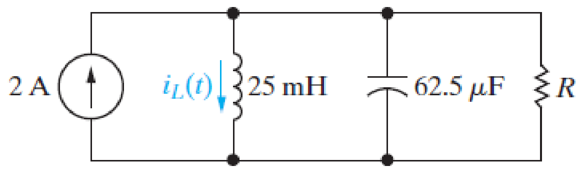
Figure P8.27



$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(25 \times 10^{-3})(62.5 \times 10^{-6})}} = 800 \text{ rad/s}$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2(12.5)(62.5 \times 10^{-6})} = 640 \text{ rad/s} \qquad \therefore \quad \text{underdamped}$$

$$\omega_d = \sqrt{800^2 - 640^2} = 480$$

$$I_f = 2 \text{ A}$$

$$i_L = 2 + B_1' e^{-640t} \cos 480t + B_2' e^{-640t} \sin 480t$$

$$i_L(0) = 2 + B_1' = 1 \qquad \text{so} \qquad B_1' = -1$$

$$\frac{di_L}{dt}(0) = -\alpha B_1' + \omega_d B_2' = \frac{V_0}{L}$$

$$\therefore \qquad -640(-1) + 480 B_2' = \frac{50}{25 \times 10^{-3}} \qquad \text{so} \qquad B_2' = 2.83$$

$$i_L(t) = 2 - e^{-640t} \cos 480t + 2.83 e^{-640t} \sin 480t \text{ A}, \quad t \geq 0$$

8.28 The resistance in the circuit in Fig. P8.27 is changed to 8 Ω. Find $i_L(t)$ for $t \geq 0$.

$$\alpha = \frac{1}{2RC} = \frac{1}{2(8)(62.5 \times 10^{-6})} = 1000 \text{ rad/s}$$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(25 \times 10^{-3})(62.5 \times 10^{-6})}} = 800 \text{ rad/s}$$

Overdamped: $\quad s_{1,2} = -1000 \pm \sqrt{1000^2 - 800^2} = -400, -1600 \text{ rad/s}$

$$I_f = 2\,\text{A}$$

$$i_L = 2 + A_1'e^{-400t} + A_2'e^{-1600t}$$

$$i_L(0) = 2 + A_1' + A_2' = 1 \qquad \text{so} \qquad A_1' + A_2' = -1$$

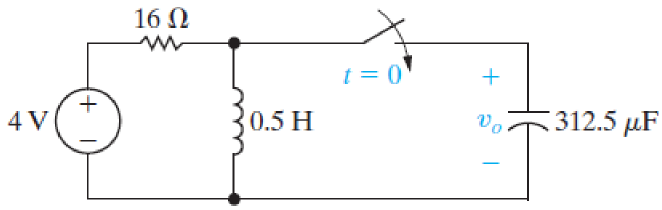
$$\frac{di_L}{dt}(0) = -400A_1' - 1600A_2' = \frac{V_0}{L} = \frac{50}{25 \times 10^{-3}} = 2000$$

Solving, $\quad A_1' = \frac{1}{3}, \qquad A_2' = -\frac{4}{3}$

$$i_L(t) = 2 + \frac{1}{3}e^{-400t} - \frac{4}{3}e^{-1600t}\,\text{A}, \quad t \geq 0$$

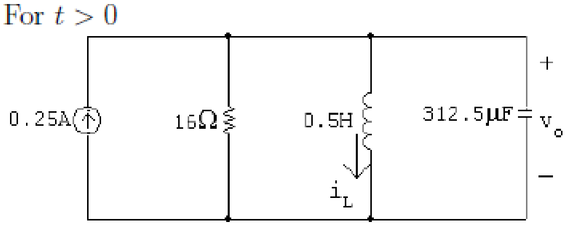
8.30 The switch in the circuit in Fig. P8.30 has been open a long time before closing at $t = 0$. At the time the switch closes, the capacitor has no stored energy. Find v_o for $t \geq 0$.

Figure P8.30



$$i_L(0^-) = i_L(0^+) = \frac{4}{16} = 0.25\,\text{A}$$

For $t > 0$



$$\alpha = \frac{1}{2RC} = 100\,\text{rad/s}; \qquad \omega_o^2 = \frac{1}{LC} = 80^2 \quad \text{so} \quad \omega_o = 80\ \text{rad/s}$$

$$s_{1,2} = -100 \pm \sqrt{100^2 - 80^2} = -40, -160\ \text{rad/s}$$

$$v_o(\infty) = 0 = V_f; \qquad A_1' + A_2' = v(0) = 0$$

$$v_o = A_1'e^{-40t} + A_2'e^{-160t}$$

$$i_C(0^+) = -0.25 + 0.25 + 0 = 0$$

$$\therefore \quad \frac{dv_o}{dt} = 0 = -40A_1' - 160A_2'$$

Solving, $\quad A_1' = 0; \qquad A_2' = 0$

$$\therefore \quad v_o = 0 \text{ for } t \geq 0$$

8.35 The switch in the circuit in Fig. P8.35 has been in the left position for a long time before moving to the right position at $t = 0$. Find

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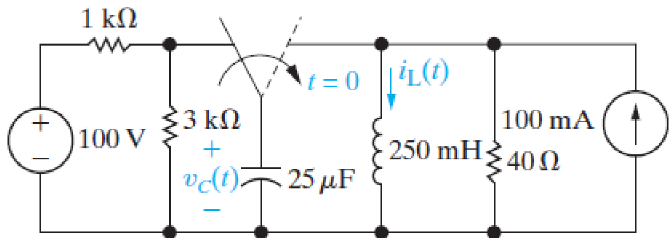
a) $i_L(t)$ for $t \geq 0$,

b) $v_C(t)$ for $t \geq 0$.

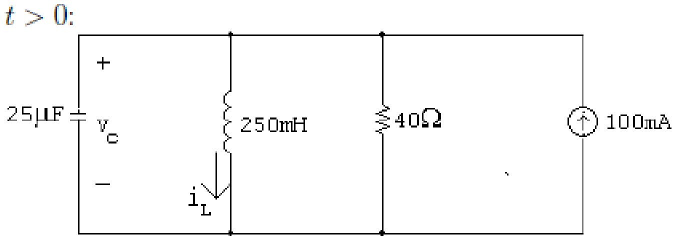


8.35 $t < 0$:

$$V_0 = v_o(0^-) = v_o(0^+) = \frac{3000}{4000}(100) = 75\,\text{V}$$

$$I_0 = i_L(0^-) = i_L(0^+) = 100\,\text{mA}$$

$t > 0$:



$$\alpha = \frac{1}{2RC} = \frac{1}{2(40)(25 \times 10^{-6})} = 500\,\text{rad/s}$$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(250 \times 10^{-3})(25 \times 10^{-6})}} = 400$$

$$\therefore \quad \alpha^2 > \omega_o^2 \quad \text{overdamped}$$

$$s_{1,2} = -500 \pm \sqrt{500^2 - 400^2} = -200, -800$$

[a] $i_L = I_f + A_1 e^{-200t} + A_2 e^{-800t}$

$$I_f = 100\,\text{mA}$$

$$i_L(0) = 0.1 + A_1 + A_2 = 0.1 \qquad \text{so} \qquad A_1 + A_2 = 0$$

$$\frac{di_L}{dt}(0) = -200A_1 - 800A_2 = \frac{V_0}{L} = \frac{75}{0.25} = 300$$

Solving, $\qquad A_1 = 0.5, \qquad A_2 = -0.5$

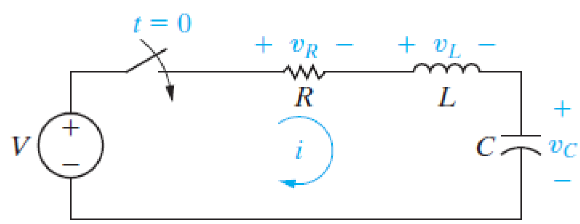
$$\therefore \qquad i_L(t) = 0.1 + 0.5e^{-200t} - 0.5e^{-800t}\,\text{A}$$

[b] $v_C(t) = v_L(t) = L\dfrac{di_L}{dt} = (0.25)(-100e^{-200t} + 400e^{-800t})$

$$= -25e^{-200t} + 100e^{-800t}\,\text{V}, \qquad t \geq 0$$

➢ Step response of a Series RLC

$$V = Ri + L\frac{di}{dt} + v_C.$$



$$i = C\frac{dv_C}{dt},$$

$$\frac{di}{dt} = C\frac{d^2v_C}{dt^2}.$$

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}.$$

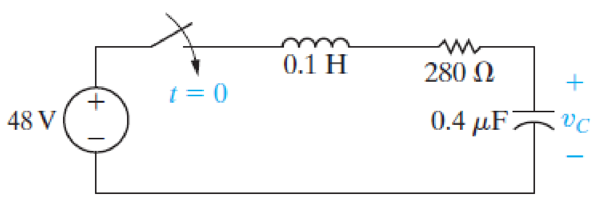
$$v_C = V_f + A_1'e^{s_1t} + A_2'e^{s_2t} \quad \text{(overdamped)},$$

$$v_C = V_f + B_1'e^{-\alpha t}\cos\omega_d t + B_2'e^{-\alpha t}\sin\omega_d t \quad \text{(underdamped)},$$

$$v_C = V_f + D_1'te^{-\alpha t} + D_2'e^{-\alpha t} \quad \text{(critically damped)},$$

Example

No energy is stored in the 100 mH inductor or the 0.4 μF capacitor when the switch in the circuit shown in Fig. 8.17 is closed. Find $v_C(t)$ for $t \geq 0$.



Solution

The roots of the characteristic equation are

$$s_1 = -\frac{280}{0.2} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$

$$= (-1400 + j4800)\ \text{rad/s},$$

$$s_2 = (-1400 - j4800)\ \text{rad/s}.$$

The roots are complex, so the voltage response is underdamped.Thus

$$v_C = V_f + B_1'e^{-\alpha t}\cos\omega_d t + B_2'e^{-\alpha t}\sin\omega_d t \quad \text{(underdamped)},$$

$$v_C(t) = 48 + B_1'e^{-1400t}\cos 4800t$$
$$+ B_2'e^{-1400t}\sin 4800t, \quad t \geq 0.$$

No energy is stored in the circuit initially, so both $v_C(0)$ and $dv_C(0^+)/dt$ are zero. Then,

$$v_C(0) = 0 = 48 + B_1',$$

$$\frac{dv_C(0^+)}{dt} = 0 = 4800B_2' - 1400B_1'.$$

Solving for B_1' and B_2' yields

$$B_1' = -48 \text{ V},$$

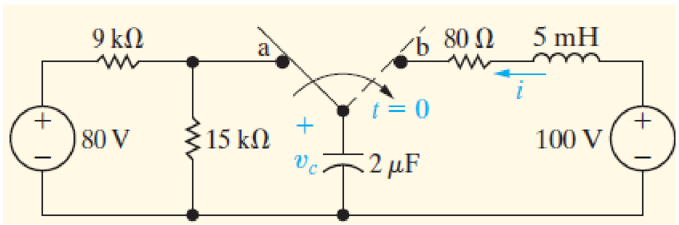
$$B_2' = -14 \text{ V}.$$

Therefore, the solution for $v_C(t)$ is

$$v_C(t) = (48 - 48e^{-1400t}\cos 4800t$$
$$- 14e^{-1400t}\sin 4800t) \text{ V}, \quad t \geq 0.$$

Q/

The switch in the circuit shown has been in position a for a long time. At $t = 0$, it moves to position b. Find (a) $i(0^+)$; (b) $v_C(0^+)$; (c) $di(0^+)/dt$; (d) s_1, s_2; and (e) $i(t)$ for $t \geq 0$.



Answer: (a) 0;
(b) 50 V;
(c) 10,000 A/s;
(d) $(-8000 + j6000)$ rad/s, $(-8000 - j6000)$ rad/s;
(e) $(1.67e^{-8000t}\sin 6000t)$ A for $t \geq 0$.

8.49 The initial energy stored in the circuit in Fig. P8.49 is zero. Find $v_o(t)$ for $t \geq 0$.

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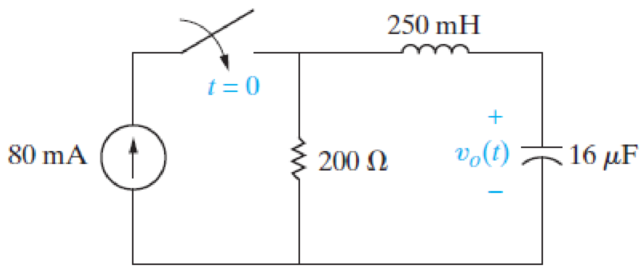
Figure P8.49



8.50 The resistor in the circuit shown in Fig. P8.49 is changed to 250 Ω. The initial energy stored is still zero. Find $v_o(t)$ for $t \geq 0$.

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8.51 The resistor in the circuit shown in Fig. P8.49 is changed to 312.5 Ω. The initial energy stored is still zero. Find $v_o(t)$ for $t \geq 0$.

8.49 $\alpha = \dfrac{R}{2L} = \dfrac{200}{2(0.025)} = 400\,\text{rad/s}$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(250\times 10^{-3})(16\times 10^{-6})}} = 500\,\text{rad/s}$$

$\alpha^2 < \omega_0^2:$ underdamped

$$\omega_d = \sqrt{500^2 - 400^2} = 300\,\text{rad/s}$$

$$v_o = V_f + B_1' e^{-400t}\cos 300t + B_2' e^{-400t}\sin 300t$$

$$v_o(\infty) = 200(0.08) = 16\,\text{V}$$

$$v_o(0) = 0 = V_f + B_1' = 0 \qquad \text{so} \qquad B_1' = -16\,\text{V}$$

$$\frac{dv_o}{dt}(0) = 0 = -400B_1' + 300B_2' \qquad \text{so} \qquad B_2' = -21.33\,\text{V}$$

$$\therefore\quad v_o(t) = 16 - 16e^{-400t}\cos 300t - 21.33e^{-400t}\sin 300t\,\text{V}, \quad t \geq 0$$

8.50 $\alpha = \dfrac{R}{2L} = \dfrac{250}{2(0.025)} = 500\,\text{rad/s}$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(250\times10^{-3})(16\times10^{-6})}} = 500\,\text{rad/s}$$

$\alpha^2 = \omega_0^2:\qquad$ critically damped

$$v_o = V_f + D_1' te^{-500t} + D_2' e^{-500t}$$

$$v_o(0) = 0 = V_f + D_2'$$

$$v_o(\infty) = (250)(0.08) = 20\,\text{V}; \qquad \therefore\quad D_2' = -20\,\text{V}$$

$$\frac{dv_o}{dt}(0) = 0 = D_1' - \alpha D_2' \qquad \text{so} \qquad D_1' = (500)(-20) = -10{,}000\ \text{V/s}$$

$$\therefore \qquad v_o(t) = 20 - 10{,}000te^{-500t} - 20e^{-500t}\,\text{V}, \quad t \ge 0$$

8.51 $\alpha = \dfrac{R}{2L} = \dfrac{312.5}{2(0.025)} = 625\,\text{rad/s}$

$$\omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(250\times10^{-3})(16\times10^{-6})}} = 500\,\text{rad/s}$$

$\alpha^2 > \omega_0^2:\qquad$ overdamped

$$s_{1,2} = -625 \pm \sqrt{625^2 - 500^2} = -250, -1000\,\text{rad/s}$$

$$v_o = V_f + A_1' e^{-250t} + A_2' e^{-1000t}$$

$$v_o(0) = 0 = V_f + A_1' + A_2'$$

$$v_o(\infty) = (312.5)(08) = 25\,\text{V}; \qquad \therefore\quad A_1' + A_2' = -25\,\text{V}$$

$$\frac{dv_o}{dt}(0) = 0 = -250A_1' - 1000A_2'$$

Solving, $\qquad A_1' = -33.33\,\text{V}; \qquad A_2' = 8.33\,\text{V}$

$$v_o(t) = 25 - 33.33e^{-250t} + 8.33e^{-1000t}\,\text{V}, \quad t \ge 0$$

The Circuit is	When	Qualitative Nature of the Response
Overdamped	$\alpha^2 > \omega_0^2$	The voltage or current approaches its final value without oscillation
Underdamped	$\alpha^2 < \omega_0^2$	The voltage or current oscillates about its final value
Critically damped	$\alpha^2 = \omega_0^2$	The voltage or current is on the verge of oscillating about its final value

Damping	Natural Response Equations	Coefficient Equations
Overdamped	$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$x(0) = A_1 + A_2$; $dx/dt(0) = A_1 s_1 + A_2 s_2$
Underdamped	$x(t) = (B_1 \cos \omega_d t + B_2 \sin \omega_d t) e^{-\alpha t}$	$x(0) = B_1$; $dx/dt(0) = -\alpha B_1 + \omega_d B_2$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Critically damped	$x(t) = (D_1 t + D_2) e^{-\alpha t}$	$x(0) = D_2$, $dx/dt(0) = D_1 - \alpha D_2$

Damping	Step Response Equations[a]	Coefficient Equations
Overdamped	$x(t) = X_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$	$x(0) = X_f + A_1' + A_2'$; $dx/dt(0) = A_1' s_1 + A_2' s_2$
Underdamped	$x(t) = X_f + (B_1' \cos \omega_d t + B_2' \sin \omega_d t) e^{-\alpha t}$	$x(0) = X_f + B_1'$; $dx/dt(0) = -\alpha B_1' + \omega_d B_2'$
Critically damped	$x(t) = X_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$	$x(0) = X_f + D_2'$; $dx/dt(0) = D_1' - \alpha D_2'$